Exercises: Green's Theorem

Problem 1. Calculate

$$\oint_C \boldsymbol{f}(\boldsymbol{r})d\boldsymbol{r}$$

where $\boldsymbol{f} = [y, -x]$, and C is the circle $x^2 + y^2 = 1$ in the positive direction.

Remark: The sign $\oint$ has the same meaning as $\int$ except that the former emphasizes that C is a *closed* curve.

Solution: Let $f_1(x, y) = y$ and $f_2(x, y) = -x$. Let D be the region enclosed by C. By Green's theorem, we know

$$\begin{aligned}
\int_C \boldsymbol{f}(\boldsymbol{r}) \cdot d\boldsymbol{r} &= \int_C (f_1\, dx + f_2 dy) \\
&= \iint_D \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\, dxdy \\
&= \iint_D -1 - 1\, dxdy = -2\pi.
\end{aligned}$$

Problem 2. Define Q as the square in $\mathbb{R}^2$ enclosing all the points (x, y) satisfying $0 \le x \le 1$ and $0 \le y \le 1$. Calculate $\oint_C \boldsymbol{f}(\boldsymbol{r})d\boldsymbol{r}$, where $\boldsymbol{f} = [6y^2, 2x - 2y^4]$, and C is the boundary of Q in the positive direction.

Solution: Let $f_1(x, y) = 6y^2$ and $f_2(x, y) = 2x - 2y^4$. Let D be the region enclosed by C. By Green's theorem, we know

$$\begin{aligned}
\int_C \boldsymbol{f}(\boldsymbol{r})d\boldsymbol{r} &= \int_C (f_1\, dx + f_2 dy) \\
&= \iint_D \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\, dxdy \\
&= \iint_D 2 - 12y\, dxdy \\
&= 2 - 12 \int_0^1 y \left(\int_0^1 dx \right) dy \\
&= 2 - 12 \int_0^1 y\, dy = -4
\end{aligned}$$

Problem 3. Calculate

$$\oint_C x^2 e^y\, dx + y^2 e^x\, dy$$

where C is the same as in the previous problem.

Solution: Let $f_1(x, y) = x^2 e^y$ and $f_2(x, y) = y^2 e^x$. Let D be the region enclosed by C. By Green's theorem, we know

$$\begin{aligned}
\int_C (f_1\,dx + f_2 dy) &= \iint_D \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\,dxdy \\
&= \iint_D y^2e^x - x^2e^y\,dxdy \\
&= \int_0^1 \left(\int_0^1 y^2e^x - x^2e^y\,dx\right) dy \\
&= \int_0^1 \left(\left(y^2e^x - \frac{e^y}{3}x^3\right)\Big|_{x=0}^{x=1}\right) dy \\
&= \int_0^1 y^2e - \frac{e^y}{3} - y^2\,dy = 0.
\end{aligned}$$

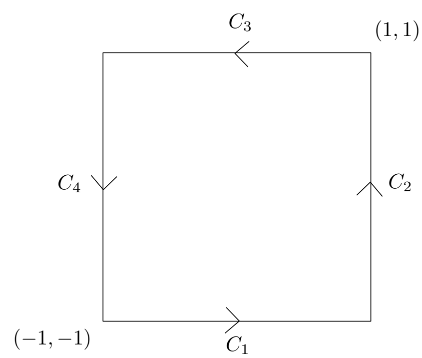
Problem 4. Define Q as the square in $\mathbb{R}^2$ enclosing all the points (x, y) satisfying $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. Calculate

$$\oint_C \left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy$$

where C is the boundary of Q in the positive direction. You can use the fact that

$$\int_{-1}^1 \frac{2}{x^2+1}\,dx = \pi.$$

Solution: Break C into four directed segments $C_1, C_2, ..., C_4$ as shown below:



$$\begin{aligned}
&\oint_C \left(\frac{-y}{x^2+y^2}\right) dx \\
= &\int_{C_1} \left(\frac{-y}{x^2+y^2}\right) dx + \int_{C_2} \left(\frac{-y}{x^2+y^2}\right) dx + \int_{C_3} \left(\frac{-y}{x^2+y^2}\right) dx + \int_{C_4} \left(\frac{-y}{x^2+y^2}\right) dx \\
= &\int_{C_1} \left(\frac{-y}{x^2+y^2}\right) dx + \int_{C_3} \left(\frac{-y}{x^2+y^2}\right) dx \\
= &\int_{-1}^1 \left(\frac{1}{x^2+1}\right) dx + \int_1^{-1} \left(\frac{-1}{x^2+1}\right) dx \\
= &\int_{-1}^1 \left(\frac{1}{x^2+1}\right) dx - \int_{-1}^1 \left(\frac{-1}{x^2+1}\right) dx \\
= &\int_{-1}^1 \left(\frac{2}{x^2+1}\right) dx = \pi.
\end{aligned}$$

Similarly:

$$\begin{aligned}
&\oint_C \left(\frac{x}{x^2+y^2}\right) dy \\
= &\int_{C_1} \left(\frac{x}{y^2+x^2}\right) dy + \int_{C_2} \left(\frac{x}{y^2+x^2}\right) dy + \int_{C_3} \left(\frac{x}{y^2+x^2}\right) dy + \int_{C_4} \left(\frac{x}{y^2+x^2}\right) dy \\
= &\int_{C_2} \left(\frac{x}{y^2+x^2}\right) dy + \int_{C_4} \left(\frac{x}{y^2+x^2}\right) dy \\
= &\int_{-1}^{1} \left(\frac{1}{y^2+1}\right) dy + \int_{1}^{-1} \left(\frac{-1}{y^2+1}\right) dy \\
= &\int_{-1}^{1} \left(\frac{1}{y^2+1}\right) dy - \int_{-1}^{1} \left(\frac{-1}{y^2+1}\right) dy \\
= &\int_{-1}^{1} \left(\frac{2}{y^2+1}\right) dy = \pi.
\end{aligned}$$

Therefore, the original integral equals 2π.

Problem 5. Prof. Goofy applies the following argument to "show" that the integral in Problem 4 equals 0. But his argument is wrong. Point out the place where he makes a mistake.

Prof. Goofy's solution: Set $f_1 = \frac{-y}{x^2+y^2}$ and $f_2 = \frac{x}{x^2+y^2}$. Thus:

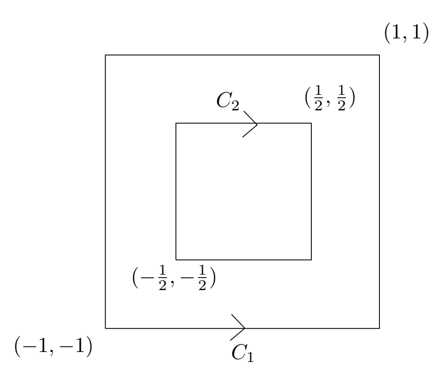
$$\begin{aligned}
\frac{\partial f_1}{\partial y} &= \frac{y^2-x^2}{(x^2+y^2)^2} \\
\frac{\partial f_2}{\partial x} &= \frac{y^2-x^2}{(x^2+y^2)^2}.
\end{aligned}$$

Let D be the area enclosed by Q. By Green's theorem, we have:

$$\oint_C \left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy = \iint_D \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} dxdy = \iint_D 0\, dxdy = 0.$$

Solution. To apply Green's theorem, the functions f_1 and f_2 need to be defined everywhere in D. This is not true: the two functions are undefined at the origin $(0, 0)$!

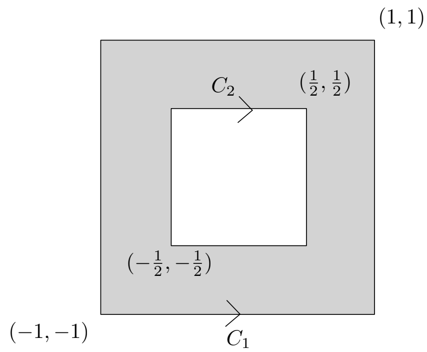
Problem 6. Suppose that C is the union of the two arcs C_1 and C_2 as shown in the following figure:



Calculate

$$\int_C (-y)\, dx + x\, dy.$$

Solution. Set $f_1 = -y$ and $f_2 = x$. Let D be the gray region as shown in the figure below:



By Green's theorem, we have:

$$\int_C (-y)\, dx + x\, dy \quad = \quad 2 \iint_D dxdy$$

which is twice the area of D, namely, 6.