Forced oscillations: beyond steady-state response

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Forced oscillations are discussed beyond the case of steady-state harmonic response at the same frequency as a harmonic driving force: transients, secular response for an undamped oscillator driven at resonance, and response to arbitrary (especially impulsive) forcing.

Contents

1 Introduction

The last module considered forced oscillation described by

$$\left(\frac{d^2}{dt^2} + 2\gamma\frac{d}{dr} + \omega_0^2\right)x(t) = \frac{F(t)}{m} \quad (1)$$

under the assumption that the force is harmonic, sAY

$$F(t) = F_0 \cos\omega t \quad (2)$$

and that the response is steady state (i.e., constant amplitude) at the same frequency, say

$$x(t) = A\cos(\omega t + \theta) \quad (3)$$

This module discusses situations that go beyond these assumptions.

- Even for this kind of forcing, there are in general transients that depend on the initial conditions. The transient solution does not have a constant amplitude, and is not characterized by the driving frequency ω.
- For an undamped oscillator driven exactly at resonance, the amplitude will grow linearly with time — a secular solution.
- More generally, the driving force may not be harmonic.

2 Transients

The steady-state solution (3) does not contain any free parameters which can be adjusted to match the initial conditions; it is only a particular solution. The parameters A and θ are determined by the ODE itself. In order to be able to match the initial conditions, we have to add a homogeneous solution, which was already discussed under free oscillations. Therefore the general solution to (1) is

$$\begin{aligned} x(t) &= A\cos(\omega t + \theta) \\ &\quad + A' e^{-\gamma t}\cos(\Omega t + \phi_0') \end{aligned} \quad (4)$$

The first term is the particular solution, with A and θ given by

$$|A|^2 = \frac{F_0^2}{m^2}\frac{1}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2} \quad (5)$$

$$\theta = -\tan^{-1}\frac{2\gamma\omega}{\omega_0^2 - \omega^2} \quad (6)$$

The second term in (4) is the homogeneous solution (see earlier module), but with the two free parameters A' and ϕ_0' denotes by $'$, and where the frequency Ω is

$$\Omega = \sqrt{\omega_0^2 - \gamma^2}$$

assuming γ is small and the homogeneous solution is under-damped. The parameters A' and ϕ_0' are to be adjusted so that (4) satisfies the initial conditions specified.

The matching of initial conditions may be messy algebraically. It seldom matters in practice, since the second term in (4) becomes negligible for $t \gg \tau = \gamma^{-1}$ — the initial conditions are "forgotten". That is why, in practice, the homogeneous solution is often ignored.

3 Secular solution

Although the homogeneous solution can often be ignored, there is one exception: if the damping is zero.

- The homogeneous solution does not decay, and cannot be ignored.
- The standard choice of the particular solution, as in (4), gives infinity if the driving frequency is on resonance.

We need a better method.

The problem is discussed both for strictly zero damping, and also for small damping — which must be closely related.

3.1 Zero damping

The system we examine is

$$\left(\frac{d^2}{dt^2} + \omega_0^2\right) x(t) = \frac{F_0}{m} \cos\omega_0 t \qquad (7)$$

There is no damping term, and the frequency is ω_0 on the RHS as well. There are many different particular solutions. We have seen that the one that is a steady-state is no good (it would have an infinite amplitude); let us look for another particular solution: the one that starts off with $x(0) = 0$ and $v(0) = 0$.

- The particle begins with zero displacement and zero velocity, but a non-zero acceleration because of the force. Thus the Taylor expansion around $t = 0$ begins with a t^2 term: $x(t) = (1/2)(F_0/m)t^2 + \ldots$. No linear combination of $\sin\omega_0 t$ and $\cos\omega_0 t$ will have this property. The solution is not harmonic.
- Since the driving frequency is exactly on resonance, it continuously pumps energy into the oscillator, so the amplitude is expected to grow linearly with time.
- So we may expect a linear combination of four types of terms:

$$\begin{array}{ccc} \cos\omega_0 t & , & t\sin\omega_0 t \\ \sin\omega_0 t & , & t\cos\omega_0 t \end{array}$$

- The system defined by the ODE and the initial conditions are symmetric in time (i.e., under $t \mapsto -t$), which eliminates the two in the second line.
- The condition $x(0) = 0$ eliminates the first one.

We are therefore left with the conjecture

$$x(t) = A\, t\, \sin\omega_0 t \qquad (8)$$

which incidentally has the property that it starts with t^2.

Problem 1
Show that with the choice

$$A = \frac{F_0}{2m\omega_0} \qquad (9)$$

then (8) satisfies (7). §

Any undamped oscillator, if driven at the resonance frequency, will have a linearly growing response, called a *secular solution.*

The general solution is therefore

$$x(t) = \frac{F_0}{2m\omega_0}\, t\, \sin\omega_0 t + x(0)\cos\omega_0 t + \frac{v(0)}{\omega_0}\sin\omega_0 t \qquad (10)$$

the second line being a homogeneous solution.

3.2 Small damping*

**This part is more advanced and can be skipped.*

We have given two mathematical descriptions for resonant driving: (a) Section 2 for $\gamma \neq 0$, and (b) Section 3.1 for $\gamma = 0$. They look quite different. Yet, the former must merge into the latter in a continuous fashion if $\gamma \to 0$. The purpose of this subsection is to demonstrate this connection. We do so for the special solution that satisfies $x(0) = 0$, $v(0) = 0$. For other cases, there is just an additional homogeneous solution, which is continuous in the $\gamma \to 0$ limit.

Start in the $\gamma \neq 0$ case and write out the general solution:

$$\begin{aligned} x(t) &= \frac{F_0}{m} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}} \cos(\omega t + \theta) \\ &\quad + e^{-\gamma t} \left(B \cos \Omega t + C \sin \Omega t\right) \end{aligned} \tag{11}$$

The first line is the harmonic particular solution, to be evaluated for $\omega = \omega_0$, and the second line is a general homogeneous solution, in which we have replaced

$$A' \cos(\Omega t + \phi_0') \quad \mapsto B \cos \Omega t + C \sin \Omega t$$

and $\Omega^2 = \omega_0^2 - \gamma^2$.

Now simplify by taking $\omega = \omega_0$ and $\gamma \to 0$, making substitutions where there are no singularities. In particular

$$\begin{aligned} \frac{1}{\sqrt{\cdots}} &\mapsto \frac{1}{2\gamma\omega_0} \\ \theta &\mapsto -\pi/2 \\ \cos(\omega t + \theta) &\mapsto \sin \omega_0 t \\ \Omega &\mapsto \omega_0 \end{aligned}$$

and we have

$$\begin{aligned} x(t) &= \frac{F_0}{m} \frac{1}{2\gamma\omega_0} \sin \omega_0 t \\ &\quad + e^{-\gamma t} \left(B \cos \omega_0 t + C \sin \omega_0 t\right) \end{aligned} \tag{12}$$

The γ in the exponential in the second line cannot be dropped, because, as we shall see, B and C may go as γ^{-1} in order to cancel the γ^{-1} in the first line.

The condition $x(0) = 0$ eliminates B, and we get

$$x(t) = \left(\frac{F_0}{2m\gamma\omega_0} + C e^{-\gamma t} \right) \sin \omega_0 t \tag{13}$$

Imposing the condition $v(0) = 0$ then gives

$$C = -\frac{F_0}{2m\gamma\omega_0}$$

and when this is put into (13) we find

$$x(t) = \frac{F_0}{2m\omega_0} \left(\frac{1 - e^{-\gamma t}}{\gamma} \right) \sin \omega_0 t \tag{14}$$

Now take $\gamma \to 0$. The expression in the brackets just gives

$$\left(\frac{1 - e^{-\gamma t}}{\gamma} \right) \to t$$

and we recover the linearly growing secular solution.

In short, if the solution is expressed in terms of the usual harmonic particular solution and a homogenous solution, then each of these go as γ^{-1} as $\gamma \to 0$. The leading singular terms must cancel, and the next term is the origin of the prefactor t.

4 Impulsive force*

*This part is more advanced and can be skipped.

4.1 Need for another method

The general problem to be solved for forced oscillations is of course (1), with an *arbitrary* driving force $F(t)$. The case of a sinusoidal driving force with a definite frequency ω, as in (2), seems to be a very special case, leaving many other cases unsolved.

There are at least two approaches for general forces. First, any force $F(t)$ can always be expressed as the sum of sinusoidal forces, e.g., if it is symmetric in time

$$F(t) = \sum_j \tilde{F}_j \cos \omega_j t$$

r in the continuous case, as the integral over such terms:

$$F(t) = \int \frac{d\omega}{2\pi} \tilde{F}(\omega) \cos \omega t$$

(The factor 2π is only a matter of convention.) If $F(t)$ is not symmetric, then $\sin \omega_j t$ or $\sin \omega t$ terms have to be added. The theorems of Fourier guarantee that every function $F(t)$ can be represented this way, as the sum or integral of sinusoidal terms.

Since we have already solved the problem for one such term, then it is only a matter of adding up the individual solutions. Or, to express the idea more physically, each frequency component can be handled separately.

This perspective is useful if the external driving force is conveniently expressed as the sum of single-frequency terms. For example, the radio waves hitting an antenna is the sum of signals from different stations, each being (centered around) a single carrier frequency. Then we simply think of each component in turn, independently.

But there are other situations in which such a representation, though theoretically possible, is by no means convenient. This Section introduces a second approach, which does not refer to sinusoidal forces (or their sums). This method is general, and especially useful for forces that lasts only a short duration, e.g., **Figure 1**.

4.2 Impulse

Reminder about impulse
Recall that an impulse is a force $F(t)$ that lasts only a short interval Δt (**Figure 2a**), during which the particle hardly moves. The impulse is the product $F(t)\Delta t$, and the only effect is that the velocity increases by an amount $\Delta v = F(t)\Delta t/m$.

Response function
Suppose the oscillator is originally at rest in the equilibrium position, and an impulse is delivered at $t = 0$, causing its velocity to jump by 1 unit. The subsequent motion is given by

$$x(t) = \psi(t)$$

where $\psi(t)$ is a solution to the homogeneous equation, satisfying the initial condition $\psi(0) = 0$, $d\psi(0)/dt = 1$.

Problem 2
Show that

$$\psi(t) = e^{-\gamma t}\,\frac{\sin\Omega t}{\Omega} \qquad (15)$$

where $\Omega^2 = \omega_0^2 - \gamma^2$. §

In fact, to write an equation that applies also before the impulse, we have

$$x(t) = \psi(t)\,\Theta(t) \equiv G(t)$$

where Θ is the unit step function, defined as

$$\Theta(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t > 0 \end{cases}$$

Single impulse at one time
Now if an impulse $F(t')\Delta t'$ is delivered at the time t', then the resultant motion must be

$$x(t) = G(t-t')\,\frac{F(t')\,\Delta t'}{m} \qquad (16)$$

where we have (a) displaced the solution by an amount t', and (b) scaled up to the size of the impulse.

4.3 General solution

Since any force $F(t)$ can be thought of as the sum of such impulses (**Figure 2b**), the general solution is obtained by adding up many terms of the form (16). Converting this to an integral, we then get

$$x(t) = \frac{1}{m}\int G(t-t')\,F(t')\,dt' \qquad (17)$$

Because of the Θ-function in G, the range of t' is $t' < t$, so we have, explicitly

$$x(t) = \frac{1}{m\Omega}\int_{-\infty}^{t} e^{-\gamma(t-t')}\,\sin\Omega(t-t') \times F(t')\,dt' \qquad (18)$$

which gives the formal solution for any force $F(t)$.

Checking the steady-state solution
The result (18) is general, and should be applicable to a harmonic driving force as well. We check the steady state solution is recovered. Replace $F(t)$ by

$$\tilde{F}(t) = F_0\,e^{i\omega t}$$

with the understanding that the result will be $\tilde{x}(t)$, whose real part is to be taken at the end. Assume this force has been present from $t' \to -\infty$; this should then produce the steady-state solution without transients. Make the substitution

$$t - t' = \tau$$

where τ has the interpretation of the delay between force and response. Then a little arithmetic leads to

$$\begin{aligned}\tilde{x}(t) &= \frac{F_0/m}{2i\Omega}\int_0^\infty e^{-\gamma\tau} \\ &\qquad \times \left(e^{i\Omega\tau} - e^{-i\Omega\tau}\right) e^{i\omega(t-\tau)}\, d\tau \\ &= \frac{F_0/m}{2i\Omega}\, e^{i\omega t}\,(I_1 - I_2) \qquad (19)\end{aligned}$$

where

$$I_{1,2} = \int_0^\infty e^{-(\gamma \mp i\Omega + i\omega)\tau}\, d\tau$$

Problem 3
Show that

$$I_1 - I_2 = \frac{2i\Omega}{(-\omega^2 + \omega_0^2) + 2i\gamma\omega}$$

and hence show that $\tilde{x}(t)$ agrees with the known steady-state solution. §

Problem 4
Suppose the force did not start from $t' \to \infty$ but say $t' = 0$. Then the upper limit in the integrals would be $\tau = t$ rather than $\tau = \infty$. Show that there is an additional term which represents the transients. There is no need to evaluate the coefficients which appear in the final formula. §

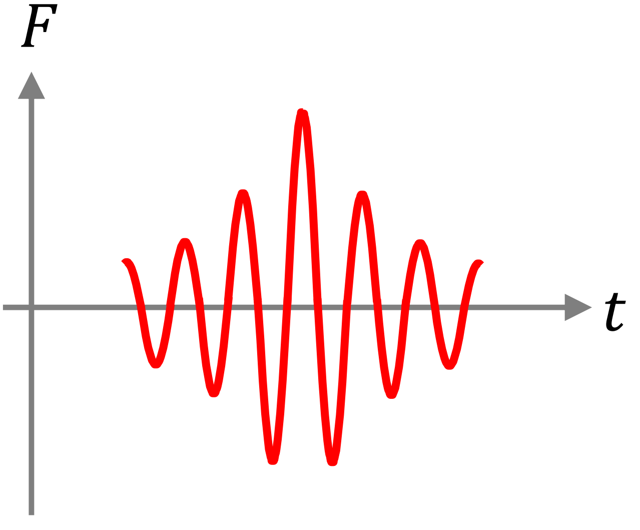

Figure 1

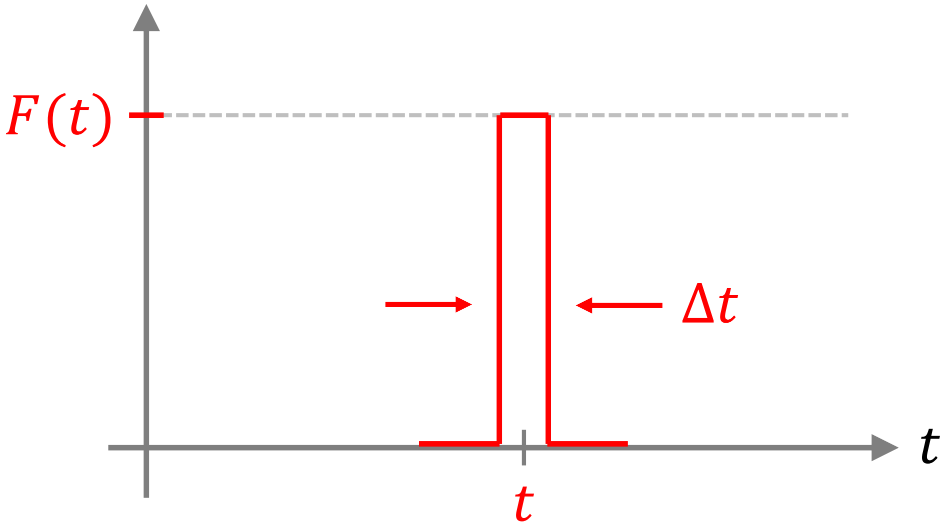

Figure 2a

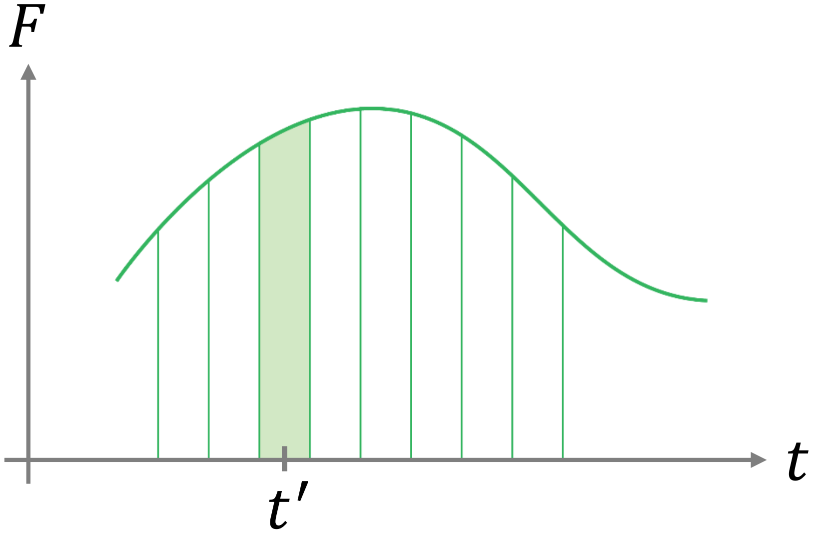

Figure 2b