Forced oscillations

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The motion of an oscillator subject to external harmonic forcing is analyzed, focusing on the steady-state response and the phenomenon of resonance. The problem is first studied without damping, and then more realistically with damping. The complex method is essential in the latter case.

Contents

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1 Introduction

The basic model

This module studies the response of an oscillator subject to an external force $F(t)$. The general equation of motion we study is

$$m\frac{d^2x}{dt^2} \quad = \quad -kx - b\,\frac{dx}{dt} + F(t)$$

where on the RHS there are (a) the restoring force due to the spring, (b) the viscous damping force, and (c) the assumed external force. This is then written in the following standard form

$$\left(\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2\right)x(t) \quad = \quad \frac{F(t)}{m} \qquad (1)$$

with all the terms proportional to x on the LHS and the inhomogeneous term on the RHS. Here $\omega_0^2 = k/m$, $2\gamma = \alpha = b/m$. The parameter γ is used (instead of α) because it is the damping rate for the amplitude of free (i.e., unforced) oscillations.

Harmonic external force

Specialize to the case where the external force is itself harmonic, say

$$F(t) \quad = \quad F_0 \cos\omega t \qquad (2)$$

about which several remarks can be made.

- Distinguish between ω_0, the *natural* frequency[1] and the *external* or *driving* frequency ω. We can likewise talk about the two relevant periods T_0 and T. Imagine say a swing which, if set into free oscillations, would execute motion with a period of $T_0 = 5$ seconds, but someone pushes it periodically with a period of $T = 3$ seconds.

- The natural frequency ω_0 is fixed for a given system (e.g., $\omega_0^2 = k/m$ if the ODE refers to a mass m tied to a spring k). But imagine that the driving frequency ω can be arbitrarily tuned, and we study how the response depends on ω as the latter is varied.

- The most important case is when ω is (approximately) equal to ω_0, which will lead to the phenomenon of *resonance.* Heuristically, if the external force pushes forward ($F > 0$) when the mass is naturally moving forward ($v > 0$) then there will be positive work done and energy is fed into the system, so one gets a large amplitude.

[1] Strictly speaking these are *angular* frequencies; the loose terminology should not lead to confusion.

- Since the system is linear, the generalization to a sum or integral of such harmonic forces with different values of ω is straightforward, and should be understood.

- In (2) we could have added a phase and written $\cos(\omega t + \psi)$. That is equivalent to a trivial shift of the origin of time. Since (at least in this module) we shall not be dealing with initial conditions, this would have no effect other than adding a similar phase to all such trigonometric functions. This will be understood to save some writing.

Further simplifications

We first deal with the *steady-state* solution, i.e., what happens after the external force has been acting for a sufficiently long time. From a physical point of view, the initial conditions will have been "forgotten" — hence the word "transients" is often used. From a mathematical point of view, this means only a particular solution is sought, while the homogeneous solution is ignored. The latter will be added back in Section **??** for the sake of completeness.

It should be noticed, however, that the homogeneous solution decays as $e^{-\gamma t} = e^{-t/\tau}$. If the damping is weak (and in the extreme case if it is zero), it will take a very long time for the initial conditions to be "forgotten". In that case the homogeneous solution must be considered as well. In the extreme case of zero damping and the driving force exactly on resonance, a somewhat different approach is called for. This case will be considered in Section **??**.

The next Section starts with the case case of no damping ($\gamma = 0$), which can be handled without invoking a complex representation. The main physical features will be introduced in this context. Then Section 3 deals with the case with damping.

2 Case without damping

2.1 The solution

In the absence of damping ($\gamma = 0$) and with the external force assumed to be harmonic as discussed above, (1) reduces to

$$\left(\frac{d^2}{dt^2} + \omega_0^2 \right) x(t) = \frac{F_0}{m} \cos \omega t \qquad (3)$$

for which a particular solution is sought to describe the steady-state response. We conjecture a solution of the form

$$x(t) = A \cos \omega t \qquad (4)$$

at the same frequency and the same phase as the external force (or half a cylce out of phase if A is negative — see below); the amplitude A is to be determined. The assumed displacement is simple harmonic motion (SHM), at the driving frequency.

For this assumed form of solution, clearly

$$\frac{d^2}{dt^2} \mapsto -\omega^2$$

so when this is put into (3), all terms have the same time dependence $\cos \omega t$, which can be cancelled, giving

$$\left(-\omega^2 + \omega_0^2 \right) A = \frac{F_0}{m}$$

thus determining the amplitude to be

$$A = \frac{F_0}{m} \frac{1}{-\omega^2 + \omega_0^2} \qquad (5)$$

- The cosine function returns to the same form after *two* differentiations.

- The original equation (3) is an identity in t, containing an infinite number of conditions, one for each t. It is now reduced to a single condition (5) on the coefficient. We are able to satisfy the original equation for all t because the conjectured form of the solution is correct.

- We neither have, not do we need, the condition that the function returns to the same form after *one* differentiation. This is one generalization needed when damping is included.

2.2 Properties of the solution

Amplitude
The most important property is that of *resonance*. The amplitude A goes as[2]

$$A^2 = \left(\frac{F_0}{m}\right)^2 \frac{1}{(\omega^2 - \omega_0^2)^2} \qquad (6)$$

which becomes very large when $\omega \approx \omega_0$, and in fact theoretically infinite when $\omega = \omega_0$. The plot of A^2 versus ω is shown in **Figure 1**. The solid line is according to (6), whereas the broken line indicates qualitatively what must happen when there is some damping — the amplitude cannot go to infinity. The modification happens only very near the resonance.

Phase
The sign of A is also interesting. Below resonance ($\omega < \omega_0$), $A > 0$, so $x(t)$ and $F(t)$ have the same sign — they are *in phase*. Above resonance ($\omega > \omega_0$), $A < 0$, so $x(t)$ and $F(t)$ have opposite signs — they are half a cycle *out of phase*: when the force pushes to the right, the mass moves to the left. This may seem slightly paradoxical, but is easily demonstrated with a mass tied to a spring.

The reference to half a cycle (or 180 degrees or π radians) can be explained more clearly as follows. Recall that SHM can be regarded as the projection of circular motion. **Figure 2** shows the vector OF with magnitude F_0 rotating at an angular velocity ω; its horizontal projection is the force $F(t)$. The vector OX has magnitude $|A|$, and also rotates at the same speed; its horizontal projection is $x(t)$. For $\omega < \omega_0$, OX is in the same direction as OF (**Figure 2a**); For $\omega > \omega_0$, OX is in the opposite direction as OF (**Figure 2b**). In the latter case, the two rotating vectors are half a cycle apart; it is a matter of convention whether we say OX is π radians ahead or behind OF. We shall see in the next Section that it is convenient to say that OX is an angle θ *ahead* of OF, with $\theta = -\pi$.

Therefore the phase relationship is shown in the plot of θ versus ω in **Figure 3**. The solid line, with a discontinuity at $\omega = \omega_0$, is the result obtained here. The broken line indicates the change when there is some damping: the dependence becomes continuous, as we shall see.

[2]Reference is made to the *square* of A in order to separate the discussion on the magnitude from the discussion on the sign.

3 Case with damping

3.1 Complex solution

When there is damping, instead of (3) we have to solve

$$\left(\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2\right) x(t) = \frac{F_0}{m}\cos\omega t \qquad (7)$$

It is no longer possible to assume a solution $x(t) \propto \cos\omega t$, since the first derivative term would then go as $\sin\omega t$. Trigonometric functions such as cosine and sine return to the same form only after *two* (or in general an even number of) differentiations. For the same property to hold for any number of differentiations, we have to go to exponential functions.

Thus the strategy is instead to first solve for a complex $\tilde{x}(t)$:

$$\left(\frac{d^2}{dt^2} + 2\gamma\frac{d}{dt} + \omega_0^2\right) \tilde{x}(t) = \frac{F_0}{m} e^{i\omega t} \qquad (8)$$

If a complex solution $\tilde{x}(t)$ is found for (8), then its real part

$$x(t) = \Re\,\tilde{x}(t)$$

would be a solution to (7), because

$$\text{RHS of (7)} = \Re\,[\text{RHS of (8)}]$$

We now try the guess

$$\begin{aligned} \tilde{x}(t) &= \tilde{A}\, e^{i\omega t} \\ \tilde{A} &= A\, e^{i\theta} \end{aligned} \qquad (9)$$

where by convention $A \geq 0$.[3] The amplitude $\tilde{A}$ is allowed to be a complex number, with magnitude A and phase θ. For a function of this form

$$\frac{d}{dt} \mapsto i\omega$$

so every term in (8) has the same time dependence $e^{i\omega t}$, which can be cancelled, leading to an *algebraic* equation

$$\left(-\omega^2 + 2i\gamma\omega + \omega_0^2\right)\tilde{A} = \frac{F_0}{m} \qquad (10)$$

which allows the complex amplitude to be found

$$\tilde{A} = \frac{F_0}{m}\frac{1}{(\omega_0^2 - \omega^2) + 2i\gamma\omega} \qquad (11)$$

[3]The parameters A and $\tilde{A}$ should not be confused with those of the same name used in describing free damped oscillations, i.e., the homogeneous solution. When both are considered, then a different and slightly more cumbersome notation would have to be adopted.

3.2 Physical solution

The solution
Write (9) as

$$\tilde{x}(t) \;=\; A\, e^{i(\omega t+\theta)} \qquad (12)$$

and take the real part

$$x(t) \;=\; A\,\cos(\omega t+\theta) \qquad (13)$$

We therefore recognize that the motion is SHM, at the driving frequency, with amplitude A, and a relative phase θ compared to the force.

Amplitude
From (11), the magnitude of A is thus given by

$$|A|^2 \;=\; \frac{F_0^2}{m^2}\,\frac{1}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2} \qquad (14)$$

The discussion is continued only for the case where γ is small. The second term in the denominator is important only when the first term is also small, i.e., only near $\omega=\omega_0$. Thus we make the approximation

$$\gamma^2\omega^2 \;\mapsto\; \gamma^2\omega_0^2$$

and

$$|A|^2 \;\approx\; \frac{F_0^2}{m^2}\,\frac{1}{(\omega_0^2-\omega^2)^2+4\gamma^2\omega_0^2} \qquad (15)$$

The denominator is minimum (and hence $|A|^2$ is maximum) when $\omega=\omega_0$, with a value $4\gamma^2\omega_0^2$. The denominator has double this value (and hence $|A|^2$ falls to half maximum) when

$$\begin{aligned}(\omega_0^2-\omega^2)^2 &= 4\gamma^2\omega_0^2\\ \omega^2-\omega_0^2 &= \pm\,2\gamma\omega_0\\ (\omega-\omega_0)(\omega+\omega_0) &= \pm\,2\gamma\omega_0\end{aligned}$$

Since the half maximum occurs near $\omega=\omega_0$, the second bracket on the LHS can be approximated as $2\omega_0$, so that we finally get

$$\omega-\omega_0 \;=\; \pm\,\gamma \qquad (16)$$

The plot of $|A|^2$ versus ω (**Figure 4**) shows how the strength of the response depends on the driving frequency. The parameter γ is seen as the *half width at half maximum* (HWHM) of this curve.

Also note that at the peak, the denominator is $4\gamma^2\omega_0^2$, so

$$|A_{\max}|^2 \;=\; \frac{F_0^2}{m^2}\,\frac{1}{4\gamma^2\omega_0^2} \propto \frac{1}{\gamma^2}$$

as indicated schematically in **Figure 4**. A small γ means a narrow and tall resonance curve.

Problem 1
Consider a case where $\omega_0=100$ and $\gamma=1$. Find the position and value of the peak ($|A|^2$ in units of F_0^2/m^2) (a) based on the exact formula (14) and (b) based on the approximate formula (15). §

Uncertainty principle
The parameter γ relates two properties.

- If such an oscillator is set into free motion, the amplitude decays as $e^{-\gamma t}$. So the oscillation lasts only a characteristic time

$$\Delta t \;\sim\; \gamma^{-1}$$

- If the oscillator is excited by an external force of frequency ω and the resonance curve (e.g., **Figure 4**) is measured, then the resonance frequency is determined to an accuracy of about

$$\Delta\omega \;\sim\; \gamma$$

  namely approximately the width of the response curve.

Therefore there is the relationship

$$\Delta t\cdot\Delta\omega \;\sim\; 1 \qquad (17)$$

which expresses an important and universal relationship between such a pair of complementary variables: to determine the frequency accurately requires that the system has a long lifetime. Of course, the accuracy can be worse than this estimate for other reasons, so (17) should in general be an inequality, with the RHS being a lower bound. The above is just one of many equivalent ways of expressing essentially the same idea.

Quality factor
Related to this are two ways of defining the *quality factor* of a resonance, denoted by Q and usually applied to cases of weak damping or narrow resonance. In terms of the variables we have introduced

$$Q \;=\; \frac{\omega_0}{2\gamma} \qquad (18)$$

Recall that in the resonance curve, $|A|^2$ drops to half the peak value at $\omega = \omega_0 \pm \gamma$, so the full width at half maximum (FWHM) or the *half power bandwidth* is 2γ. Therefore one interpretation is

$$Q = \frac{\text{resonance frequency}}{\text{half power bandwidth}}$$

where the numerator and denominator can be both expressed in terms of frequency f or angular frequency ω (in the latter case the FWHM is 2γ).

Secondly we know that for free oscillations, the energy decreases as

$$\mathcal{E} = \mathcal{E}_0 \, e^{-2\gamma t}$$

(apart from small oscillations with zero mean) so the energy loss in a period T is

$$\begin{aligned} \Delta\mathcal{E} &= \mathcal{E}_0 \left(1 - e^{-2\gamma T}\right) \\ &\approx \mathcal{E}_0 \, (2\gamma T) \\ &= \mathcal{E}_0 \, 2\pi \, \frac{2\gamma}{\omega_0} \end{aligned}$$

the approximate expression being valid for weak damping. Thus we see

$$Q = 2\pi \, \frac{\text{energy stored}}{\text{energy lost per cycle}}$$

Phase

From (13), θ is the angle by which the displacement $x(t)$ leads the force $F(t)$: if $\theta > 0$, then $x(t)$ leads; if $\theta < 0$, then $x(t)$ lags. In fact, the pair of variables

$$\begin{aligned} F(t) &= F_0 \, \cos \omega t \\ x(t) &= A \, \cos(\omega t + \theta) \end{aligned}$$

can be represented graphically as in **Figure 5**. They are the horizontal projections respectively of the vectors OF and OX, which are rotating together at angular velocity ω. The vector OF makes an angle ωt with the horizontal axis (and starts out along this axis at $t = 0$). The vector OX is ahead if $\theta > 0$ (**Figure 5a**) and behind if $\theta < 0$ (**Figure 5b**).

Thus the complex amplitude $\tilde{A}$ nicely captures both the magnitude and the phase of the response.

To see how θ depends on the frequency, note that from (11)

$$\begin{aligned} e^{i\theta} &\propto \frac{1}{(\omega_0^2 - \omega^2) + 2i\gamma\omega} \\ e^{-i\theta} &\propto (\omega_0^2 - \omega^2) + 2i\gamma\omega \\ -\tan\theta &= \frac{2\gamma\omega}{\omega_0^2 - \omega^2} \end{aligned}$$

or, explicitly

$$\theta = -\tan^{-1} \frac{2\gamma\omega}{\omega_0^2 - \omega^2} \tag{19}$$

This is shown versus ω in **Figure 6**.

- At $\omega = 0$, the RHS of (19) is zero, so $\theta = 0$. The displacement and the force are in phase. (Be careful: arctan has two solutions; θ can be either 0 or π. But if we go back to $e^{-i\theta}$, we see that for $\omega \to 0$, the real part is positive and the imaginary part is zero; so $\theta = 0$.)
- Now suppose ω is small; then the RHS of (19) is negative and small; so θ is negative and small. (There are always two solutions for arctan, differing by π, but continuity from $\omega = 0$ fixes the choice.)
- At $\omega = \omega_0$, the tangent is infinite, and $\theta = -\pi/2$.
- For $\omega > \omega_0$, θ lies between $-\pi/2$ and $-\pi$.
- For extremely high frequencies, the tangent goes to zero again, and $\theta \to -\pi$. By the way, this explains the convention that $\theta = -\pi$ for the undamped case above resonance.

Thus, resonance can also be specified by the condition that the response is exactly a quarter cycle out of phase. The width of the transition region can be defined by the phase being 1/8 cycle ($\pi/4$ radians or 45 degrees) away from the resonance (**Figure 6**), at which points

$$\frac{2\gamma\omega}{\omega_0^2 - \omega^2} = \pm 1 \tag{20}$$

Problem 2

Using the same approximation of small γ as before, show that the boundaries of the transition region are

$$\omega - \omega_0 = \pm \gamma \tag{21}$$

namely the same as the points for half-maximum for $|A|^2$. §

Problem 3
Go back to the ODE (1) with the force given by (2). Assume a solution of the form (13). Without invoking any complex methods, directly check that this satisfies the ODE, and determine A and θ. Hint: Write $\cos(\omega t+\theta)$ in terms of $\cos\omega t$ and $\sin\omega t$. These two types of terms must balance in the ODE, leading to two conditions on the parameters. The complex method is just a way of combining the two real conditions into one equation. §

3.3 Comparison with undamped case

The amplitude and phase are again plotted versus ω in **Figure 7**, with the corresponding undamped case shown by the broken lines. The two are approximately the same away from the resonance. This can be understood, for example, from (15). If $\omega^2-\omega_0^2$ is not small, the second term in the denominator can be neglected, i.e., setting $\gamma=0$ makes no difference. Thus, damping is important only in a band of width $\sim\gamma$ around the resonance, where it has the effect of (a) limiting the maximum amplitude to a finite value $|A| \propto 1/\gamma$, and (b) showing that the phase goes from 0 to $-\pi$ in a continuous manner over a width $\sim\gamma$, rather than discontinuously.

3.4 Work done by external force

We should also explain that a phase lag of $\pi/2$ corresponds to maximum work done (quite apart from the variation of the amplitude $A=A(\omega)$). To see this, start with

$$\begin{aligned} F(t) &= F_0\cos\omega t \\ x(t) &= A\cos(\omega t+\theta) \\ v(t) &= -A\omega\sin(\omega t+\theta) \end{aligned}$$

The rate of doing work, or the power P, is given by

$$\begin{aligned} & P = Fv \\ = & -F_0 A\omega\cos\omega t\,(\sin\omega t\,\cos\theta+\cos\omega t\,\sin\theta) \end{aligned}$$

Consider the average over a cycle, denoted by $\bar{\ }$, and note that

$$\cos^2\omega t \mapsto 1/2 \qquad , \qquad \sin\omega t\cos\omega t \mapsto 0$$

so

$$\bar{P} = \frac{1}{2}F_0 A\omega\,(-\sin\theta) \tag{22}$$

The amplitude A depends on ω. Setting this aside, let us look at the dependence on the phase, through the factor in brackets.

- Since $0 \geq \theta \geq -\pi$, the bracket is non-negative.
- This means the external force (on average) delivers energy to the oscillator, not the other way round — which is intuitively obvious.
- There is (on average) no work done if the phase angle is $\theta=0$ or $\theta=-\pi$. (The amplitude is also negligible in these cases.)
- This factor is maximum at resonance, when $\theta=-\pi/2$. At this point, the force and the velocity are in phase.

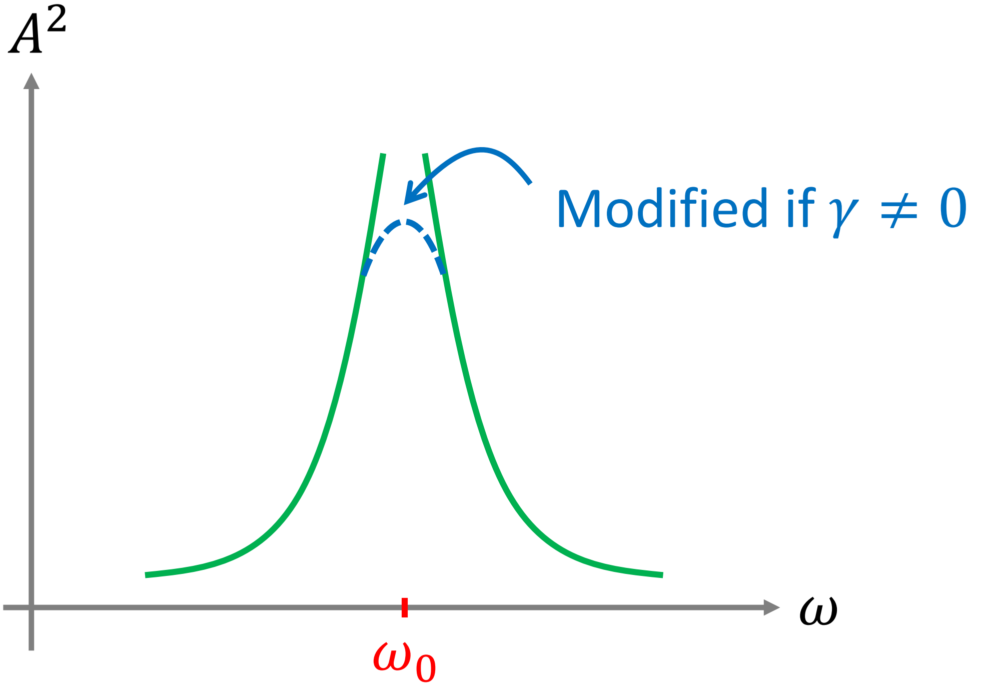

Figure 1

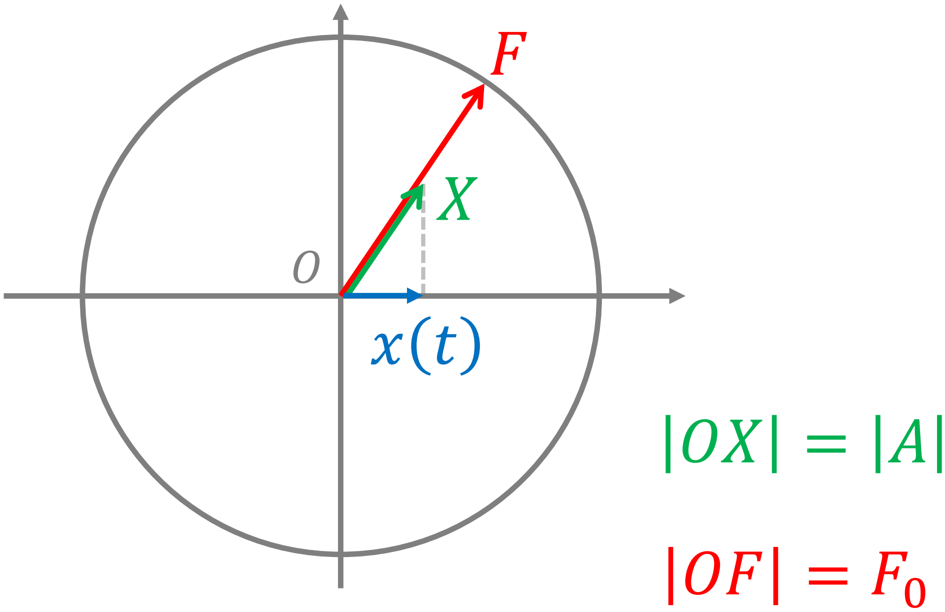

Figure 2a

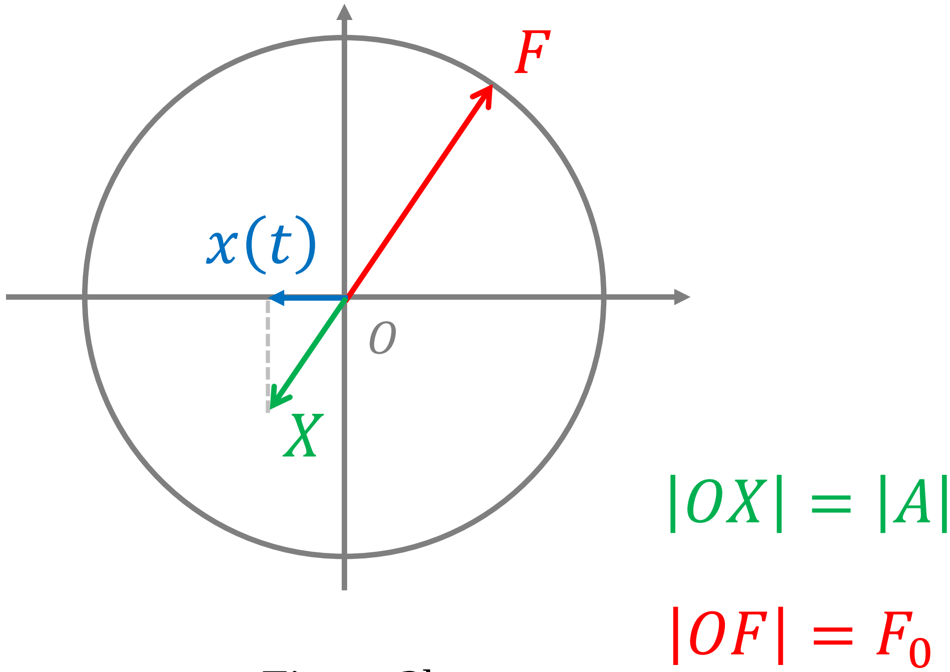

Figure 2b

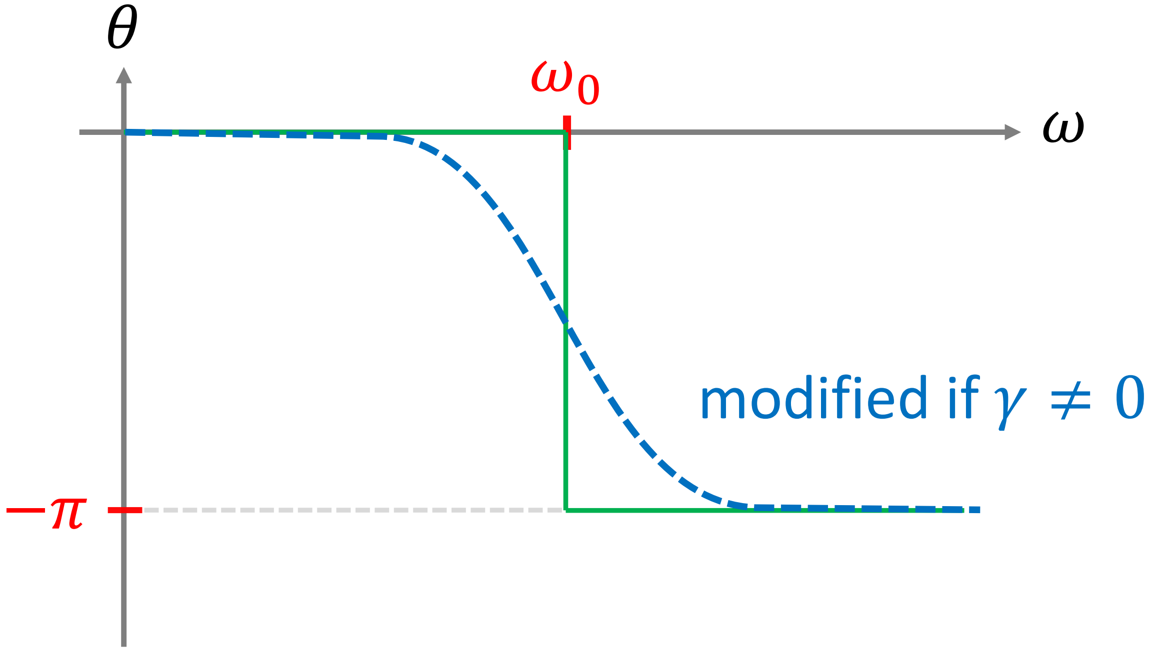

Figure 3

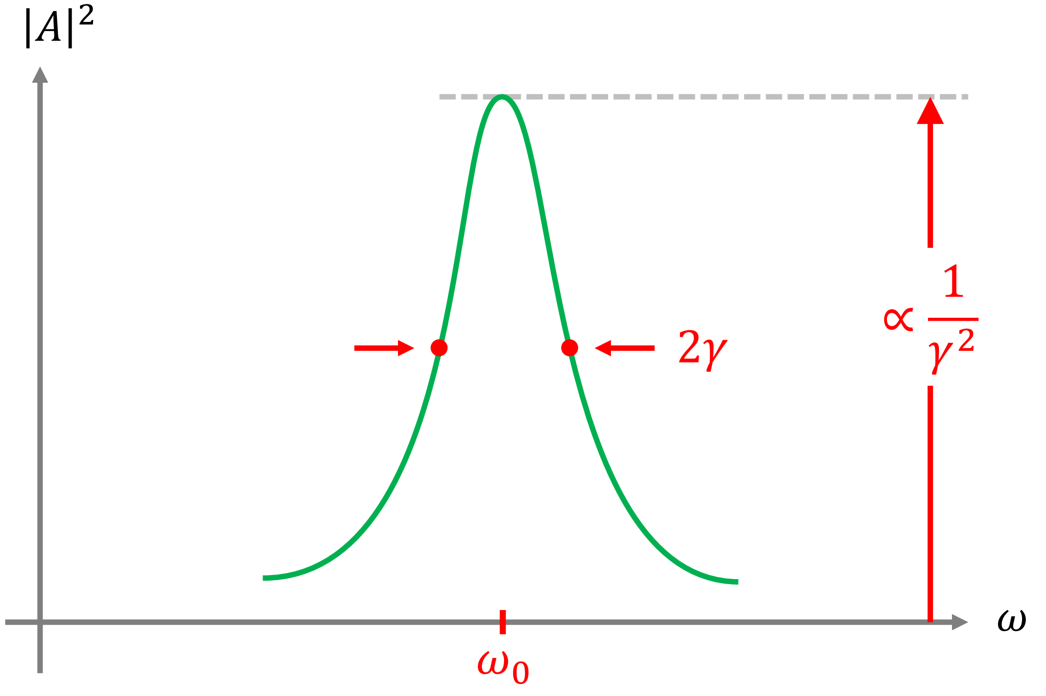

Figure 4

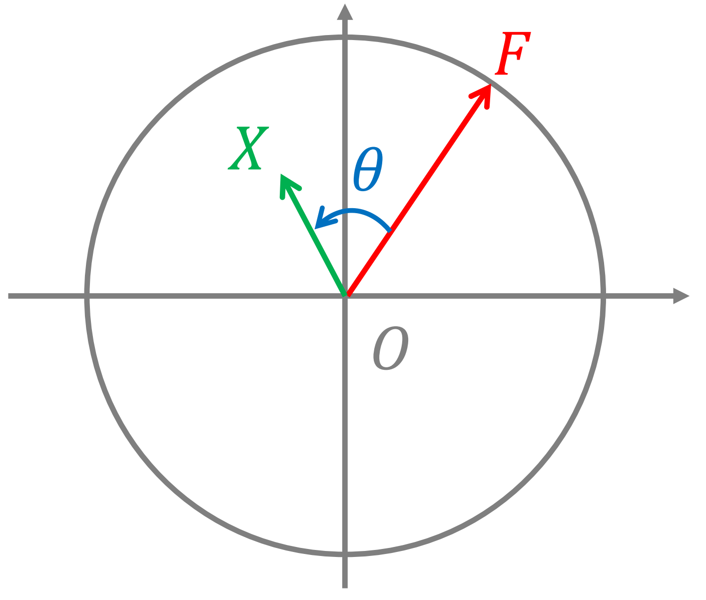

Figure 5a

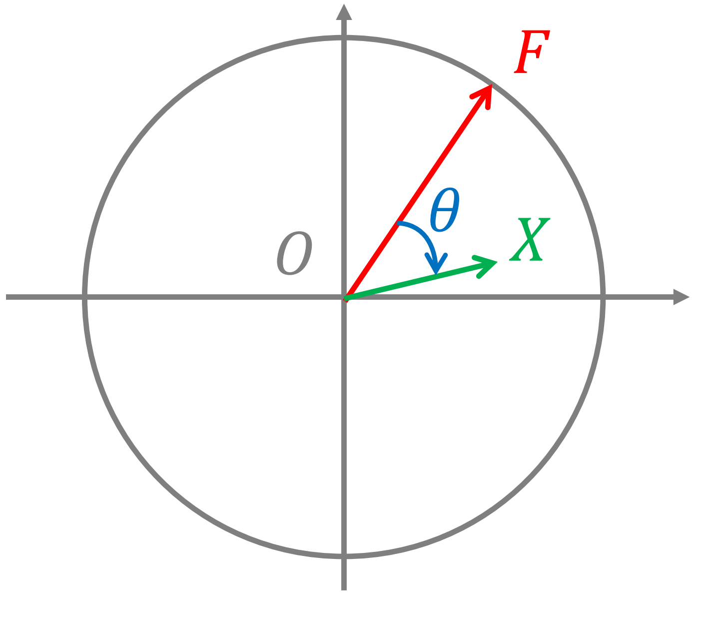

Figure 5b

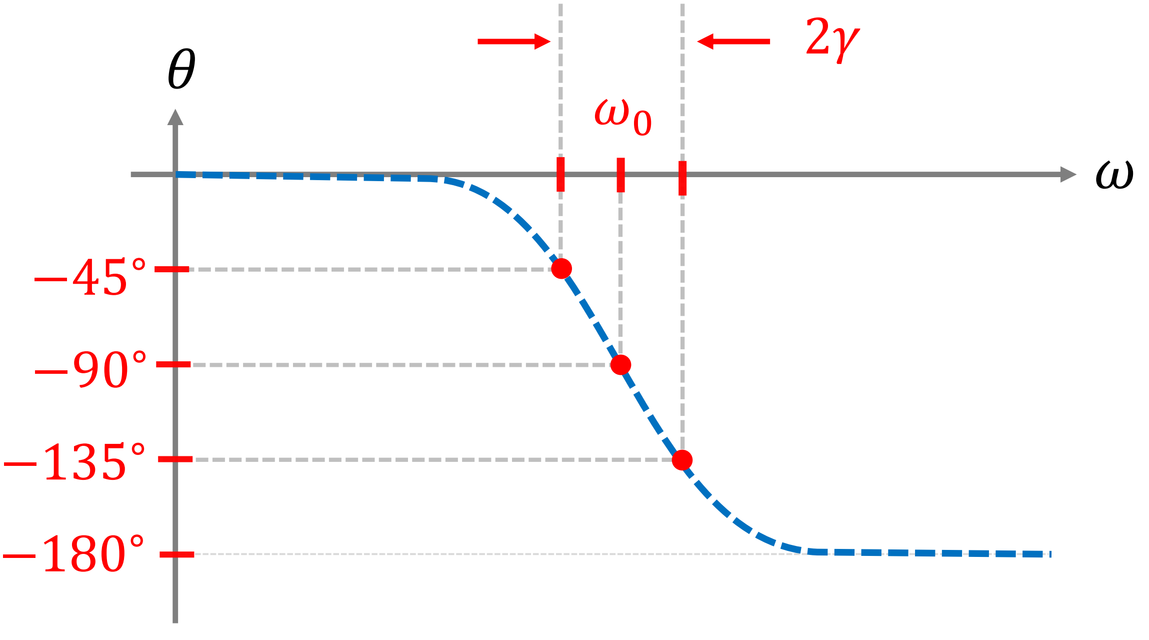

Figure 6

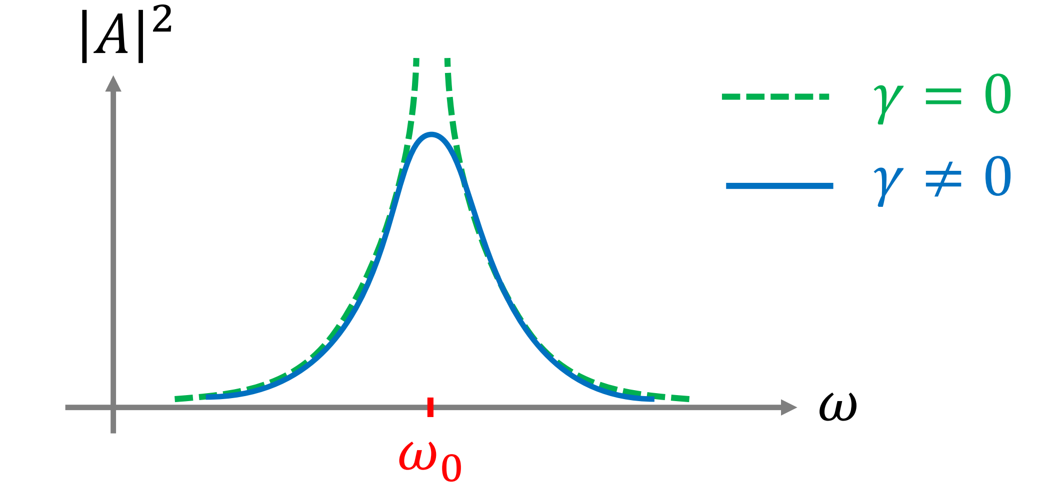

Figure 7a

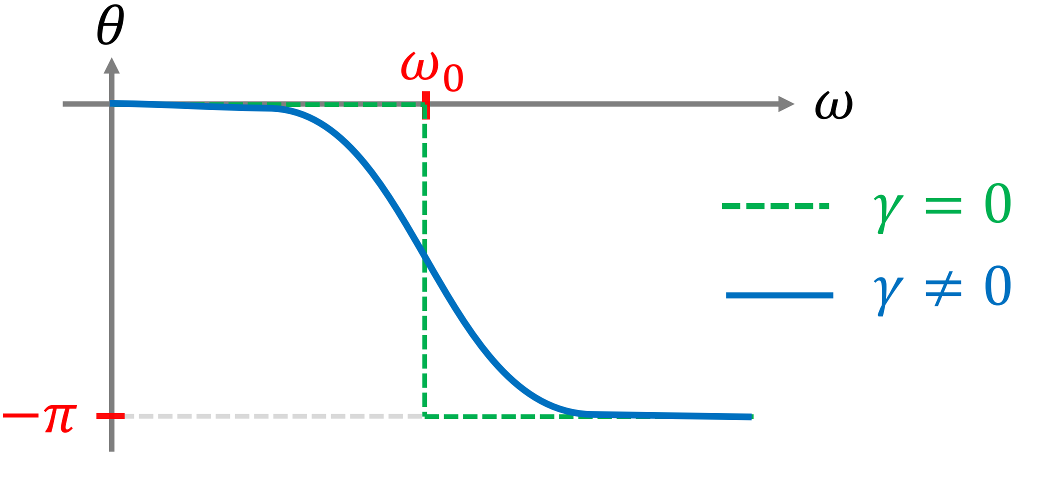

Figure 7b