Ordinary differential equations

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In physics, one tries to determine the motion of a particle from the forces. The corresponding problem in mathematics is to solve an ordinary differential equation. Some general methods and theorems are presented in a more systematic way, with particular attention on equations that (a) are linear and especially (b) have constant coefficients.

Contents

1 Introduction

Mechanics leads to ODEs
In mechanics, one tries to solve for the position $x(t)$ of a particle from the forces and the given initial conditions. If the net force F is given in terms of position, velocity and perhaps also time: $F = F(x, v, t)$, then from Newton's second law, the problem becomes one of solving

$$m\,\frac{d^2x}{dt^2} \;=\; F(x, \dot{x}, t) \qquad (1)$$

which is an *ordinary differential equation* or ODE. The example of SHM is a particularly simple one, in which F is (a) independent of $\dot{x}$ and t, and (b) linear in x. At least in that simple case, we have already found the solution. The purpose of this module is to discuss ODEs more generally.

Notation
We shall freely alternate between the two sets of notations

$$(x, v, a) \;\equiv\; (x, \dot{x}, \ddot{x})$$

Order
The *order* of an ODE is the highest degree of derivative that appears. Most ODEs arising from physics are second order, as in (1). We shall mostly pay attention to second-order ODEs, though some results below are more general.

Types
There are many ways to classify ODEs. In this module we consider the following categories, whose mutual relationships are illustrated in **Figure 1**:

- $\mathcal{A}$ = all ODEs
- $\mathcal{B}$ = linear ODEs
- $\mathcal{C}$ = linear ODEs with constant coefficients

Examples will be given below, and we shall also consider $\mathcal{B}'$ and $\mathcal{C}'$ which are ODEs that contain an inhomogeneous term.

Mathematics and physics
This module is mathematical, and should be cross referenced to physical examples and concepts. The more elementary parts relate to the previous module in an obvious way. The more advanced topics should be revisited after studying damped and forced oscillations.

2 Numerical method

2.1 The method

Go back to the example of SHM and write it in the form

$$a = -\omega_0^2 x$$

It is given that $\omega_0^2 = 4.0$; the initial position is $x(0) = 0.03$ and the initial velocity is $v(0) = 0.08$. (Henceforth, MKS units are understood and not displayed.) Pretend we do not know the analytic solution. There is another way to solve the problem — numerically. We need to find find $x(t)$ satisfying

$$\begin{aligned} x(0) &= 0.030000 \\ v(0) &= 0.080000 \\ a(t) &= -4.0 \times x(t) \end{aligned}$$

Chop the time into short intervals of $\Delta t = 0.1$; we assume *this is small enough that within each interval, the acceleration can be regarded as uniform.* Then it is straightforward, using the formulas for uniform acceleration, to move forward one little step at a time.

$$\begin{aligned} a(0) &= -4.0 \times x(0) = -0.120000 \\ v(0.1) &= v(0) + a(0)\Delta t \\ &= 0.080000 + (-0.120 \times 0.1) \\ &= 0.068000 \\ x(0.1) &= x(0) + (1/2)[v(0) + v(0.1)]\Delta t \\ &= 0.030000 + (1/2)(0.080 + 0.068) \times 0.1 \\ &= 0.0374000 \end{aligned}$$

Because the time step is small, the change of any variable in one time step is also small; in order for these changes to be accurately captured, a fairly large number of digits have been kept — probably more than is necessary.

Problem 1
Continue the calculation for one more interval. §

Problem 2
Set up a table with columns representing t, x, v, a, and put the results of Problem 1 into the table. Continue the table for three more rows. §

Short time interval
The *only* assumption is that the time interval Δt is small enough that the acceleration can be regarded as uniform within each interval. How do we know the choice of $\Delta t = 0.1$ is good enough?

- The period is $T \sim 3$. (This is known from the analytic result, with $\omega_0 = 2$, or approximately after following the numerical solution through one cycle.) So the time step of $\Delta t = 0.1$ is $\sim$ 0.03 of a period, which should not be too bad. In other words, the comparison must refer to a characteristic time scale of the problem.

- A better answer is as follows. After solving the problem with one value of Δt, repeat it for a smaller value, say half (which would then involve twice as many steps). Check that the two answers agree to the accuracy desired.

Using a spreadsheet
For a sufficiently small Δt, many steps are needed. It is best to automate the calculation by putting the table of Problem 2 onto a spreadsheet. Formulas need to be entered only once, and then extended to other intervals using COPY and PASTE. The spreadsheet `eqm.xlsx` illustrates the above procedure in three ways.

- Sheet 1 shows the calculation as above.

- Sheet 2 repeats it for a smaller (and adjustable) Δt, and can be used to check the convergence as $\Delta t \to 0$. In this case, we have chosen a different set of initial conditions.

- Sheet 3 (which is laid out more systematically) generalizes to the equation of motion

  $$a = -(c_1 x + c_3 x^3)$$

  for arbitrary c_1 and c_3. (The spreadsheet shows the case for one set of (c_1, c_3), but these parameters are easily changed.)

If you look at the situation after one cycle, you will find that the amplitude has changed slightly; this is an indication of the size of numerical error.

Our purpose is just to illustrate the idea. There are higher-order algorithms which give better accuracy for the same Δt. Moreover, there are more efficient softwares (e.g., using FORTRAN or C++) than spreadsheets, and you will learn these in other courses.

Problem 3
An oscillator obeys the force law $a = -x^3$.
(a) Using the spreadsheet provided, find the period of motion for amplitude $A = 1.0$. (Start with $x(0) = 1.0$, $v(0) = 0$ and continue until $x = 0$; the time is $T/4$.)
(b) Repeat for $A = 2.0$ and check the dependence against what you expect from dimensional analysis.
§

2.2 Why numerical method

We already have an analytic solution for SHM; so why bother with a numerical solution, which for any finite Δt is not even exact?

- The numerical method is applicable for *any* force law $a = f(x, v)$; sheet 3 and Problem 3 illustrate one case. For most such force laws, there is no analytic solution.
- The numerical method[1] shows that a solution exists if two initial conditions $x(0)$ and $v(0)$ are given. (Existence)
- The numerical method also shows that with these initial conditions, there is only one solution. (Uniqueness)

Uniqueness
Since the solution given the initial condition is unique, we are allowed to *guess* the solution: so long as we check that it satisfies the equation, then it is *the* correct solution. In the earlier discussion of SHM, we guessed the solution; now we have the formal justification.

Number of free parameters
Moreover, the general solution to a second-order differential equation must contain *two free parameters* in order to match the two initial conditions. (More generally, for an n-th order ODE, there should be n free parameters to match n initial conditions.)

[1]Subject to niceties about the existence of the limit $\Delta t \to 0$, which we leave to mathematicians to worry about

3 Linearity and superposition

In general, not much can be said about the properties of most ODEs. For example, if a particle is subject to a viscous force proportional to the *third power* of the velocity, and to a potential energy, say $U(x) = U_0 \cos kx$, then the equation of motion would be

$$m\frac{d^2x}{dt^2} = -b\left(\frac{dx}{dt}\right)^3 + kU_0 \sin kx$$

about which not much can be said. However, there are useful theorems which apply if the ODE is *linear* (class $\mathcal{B}$), and in particular if the ODE is *linear with constant coefficients* (class $\mathcal{C}$).

3.1 Linear

Consider a differential operator of the form

$$D = a_n(t)\frac{d^n}{dt^n} + \ldots + a_1(t)\frac{d}{dt} + a_0(t) \qquad (2)$$

(with $a_n(t) \neq 0$). Such an operator is said to be *linear* in the sense that

$$\begin{aligned} & D\left[c_1x_1(t) + c_2x_2(t)\right] \\ = \; & c_1\left[D\,x_1(t)\right] + c_2\left[D\,x_2(t)\right] \end{aligned}$$

and the corresponding equation $D\,x(t) = 0$ is said to be a *linear ODE*; this class of ODEs was denoted as $\mathcal{B}$.

3.2 Superposition

For a linear ODE, if there are two solutions $x_1(t)$ and $x_2(t)$, i.e.,

$$D\,x_1(t) = 0 \quad , \quad D\,x_2(t) = 0$$

then any linear combination with constant coefficients c_1, c_2

$$x(t) = c_1x_1(t) + c_2x_2(t)$$

is also a solution, i.e.,

$$D\,x(t) = 0$$

In physics, we typically deal with second-order ODEs, for which we can (and must) specify two initial conditions, which would give a unique solution. This means there can be two and only two independent solutions only (say $x_1(t)$ and $x_2(t)$), and the two corresponding coefficients c_1 and c_2 would allow the two initial conditions to be satisfied. If there were a third independent solution, that would make the solution not unique — and that contradicts what we learnt from the numerical method.

3.3 Real equation and complex solution

Consider an equation $D\,x(t) = 0$ with D given by (2) and all the coefficients $a_j(t)$ being real, $j = 0, \ldots, n$, as would usually be the case if the equation arises from physics. We say that the ODE is a real equation.

But a real equation can have complex solutions. (This happens even in algebra, where $ax^2 + 2bx + c = 0$ with real a, b, c can have complex solutions if $b^2 - ac < 0$.) Let $\tilde{x}(t)$ be such a complex solution:

$$D\,\tilde{x}(t) = 0$$

Take the complex conjugate and note that $D^* = D$, and we find that

$$D\,\tilde{x}^*(t) = 0$$

namely that the conjugate function is also a solution.

Then by the superposition principle, the following are also solutions

$$\Re\,\tilde{x}(t) = \frac{1}{2}\left[\tilde{x}(t) + \tilde{x}^*(t)\right]$$
$$\Im\,\tilde{x}(t) = \frac{1}{2i}\left[\tilde{x}(t) - \tilde{x}^*(t)\right]$$

So from one complex solution we can generate two real solutions.

It will be seen, especially in the next Section, that it is often convenient to first seek complex solutions.

3.4 Linear with constant coefficients

There is a special subclass, for which the coefficients a_j are constants independent of t:

$$D = a_n \frac{d^n}{dt^n} + \ldots + a_1 \frac{d}{dt} + a_0 \qquad (3)$$

This class of ODEs was denoted as $\mathcal{C}$. A general method for solving such equations is discussed in the next Section.

SHM

For the case of SHM, we have $n = 2$ and in particular:

$$D = \frac{d^2}{dt^2} + \omega_0^2$$

In this case, even without a general method, we can guess two solutions:

$$x_1(t) = \cos\omega_0 t$$
$$x_2(t) = \sin\omega_0 t$$

which are easily checked. Thus the general solution is given by a linear superposition, namely

$$x(t) = B\cos\omega_0 t + C\sin\omega_0 t \qquad (4)$$

as obtained before.

Next we outline a more systematic method to deal with such ODEs with constant coefficients.

4 ODEs with constant coefficients

4.1 The characteristic equation

Consider the equation $D\,x(t) = 0$ with D given by (3), and let us *guess* a (possibly complex) solution

$$\tilde{x}(t) = e^{i\omega t}$$

This function has the nice property that each differentiation just gives a multiplicative factor:

$$\frac{d}{dt}\,\tilde{x}(t) = i\omega\,\tilde{x}(t)$$
$$\frac{d}{dt} \mapsto i\omega$$

When this is put into (3), we find an *algebraic* condition, that ω must be a root of the *characteristic polynomial*:

$$\tilde{D}(\omega) \quad = \quad a_n(i\omega)^n + \ldots + a_1(i\omega) + a_0 \qquad (5)$$

Such a polynomial equation is guaranteed to have n roots, which we denote as $\omega_1, \ldots, \omega_n$.

- If $n = 1$ or $n = 2$, $i\omega_j$ can be found analytically. For $n > 2$ (even though there are analytic methods for $n = 3, 4$), the solutions can be found numerically, by Newton's algorithm for example.

- We could have chosen the conjectured solution to be

$$\tilde{x}(t) \quad = \quad e^{\beta t}$$

  i,e., replace $i\omega \mapsto \beta$. In general, we do not expect the roots of the characteristic polynomial to be either purely real or purely imaginary. Thus, neither β nor ω should be thought of as necessarily real. Which form we use is purely a matter of convenience and convention. The $i\omega$ notation is usually adopted if the system is expected to be (mostly) oscillatory.

- If the coefficients a_j are real, i.e., (5) is a real polynomial, then the solutions for $\beta = i\omega$ are either real, or are in complex-conjugate pairs.

- The case where two (or more) roots of (5) coincide would be ignored for now. The mathematical subtlety can be bypassed: We can always change the coefficients a_j slightly to split the solutions, and then take the limit to remove the splitting.

4.2 The general solution

Assuming ω_j to have been found, then by superposition, a general complex solution is

$$\tilde{x}(t) \quad = \quad \sum_j \tilde{A}_j \, e^{i\omega t}$$

where

$$\tilde{A}_j \quad = \quad A_j \, e^{i\phi_j}$$

are arbitrary complex amplitudes associated with the various terms.

Since the values $i\omega_j$ are not necessarily pure imaginary, we write them as

$$\omega_j \quad = \quad \Omega_j + i\gamma_j \qquad (6)$$

Thus we can also write (**??**) as

$$\tilde{x}(t) \quad = \quad \sum_j A_j \, e^{-\gamma_j t} \, e^{i(\Omega_j t + \phi_j)} \qquad (7)$$

4.3 The general real solution

From this point onwards, assume a_j are real. By taking the real part of (7), we get a real solution

$$x(t) \quad = \quad \sum_j A_j \, e^{-\gamma_j t} \, \cos(\Omega_j t + \phi_j) \qquad (8)$$

This is the general real solution. Each term represents a harmonic motion whose amplitude is decreasing with time as

$$A_j(t) \quad = \quad A_j \, e^{-\gamma_j t} \qquad (9)$$

As a matter of mathematics, γ_j can have either sign; but most applications in physics would involve a decreasing solution, hence the convention.

It may look as if there are $2n$ free parameters (A_j, ϕ_j), whereas we know that we can only specify n initial conditions. This mystery is left as an exercise. It is best to consider the case $n = 2$ and the various possibilities for the two roots of the characteristic equation.

It is important to note, from (6), that the real part of ω_j represents oscillation, and the imaginary part represents damping.

5 Inhomogeneous case

5.1 Defining the problem

We now make a small generalization: Consider the equation

$$D \, x(t) = f(t) \qquad (10)$$

with D given by (2). Such an ODE is said to be *inhomogeneous*.

For the moment we do not assume the coefficients $a_j(t)$ to be time-independent. We want to solve this ODE with suitable initial conditions, in general with given values of $x, dx/dt, \ldots, d^{n-1}x/dt^{n-1}$ at $t = 0$.

5.2 Splitting into two parts

Write

$$x(t) \quad = \quad x_h(t) + x_p(t) \tag{11}$$

where $x_p(t)$ is any one *particular solution* that satisfies (10). The name "particular" means that it does not contain any free parameters that would allow the initial conditions to be matched.

It is easy to check that $x(t)$ would satisfy (10) if the other term, called the *homogeneous solution*, satisfies

$$D\, x_h(t) \quad = \quad 0 \tag{12}$$

Thus the second part of the problem is reduced to the homogeneous case, for which the solution is assumed known. In particular, x_h would contain n free parameters to allow matching of the initial conditions.

We next consider three classes of inhomogeneities for which the particular solution can be solved relatively simply.

5.3 Constant coefficients and constant inhomogeneity

This idea is readily illustrated in the special case where a_n are independent of time, and $f(t)$ is also independent of time. Then a particular solution is simply

$$x_p(t) \quad = \quad f/a_0$$

while the general homogenous solution is given by (8), so now we have

$$\begin{aligned} & x(t) \\ = \quad & \sum_j A_j\, e^{-\gamma_j t} \cos(\Omega_j t + \phi_j) + f/a_0 \end{aligned} \tag{13}$$

An oscillating system subject to an extra constant external force is exactly an example of what is described here. The extra constant term in the solution in that case corresponds simply to measuring the displacement from the new equilibrium position.

5.4 Harmonic inhomegeneity

Again consider the case where the coefficients $a_j(t)$ are time-independent, and suppose the inhomogeneous term on the RHS of (10) is harmonic. We consider the corresponding complex problem:

$$\begin{aligned} D\,\tilde{x}(t) \quad &= \quad \tilde{f}(t) \\ \tilde{f}(t) \quad &= \quad \tilde{f}_0\, e^{i\omega t} \end{aligned} \tag{14}$$

- The parameter ω has nothing to do with any of the ω_j found from the characteristic equation. It is simply the frequency of the inhomogeneous term, or, to take a physical example, the frequency of an external force driving an oscillator.
- In physics we typically deal with *real* $f(t)$ and $x(t)$, in which case the real part of (14) is to be taken.

We *guess* a solution

$$\tilde{x}(t) \quad = \quad \tilde{x}_0\, e^{i\omega t} \tag{15}$$

Now (unlike the case when we were solving the homogeneous equation), ω is not a free parameter to be determined, but the same ω as in the inhomogeneous term; rather, the complex amplitude $\tilde{x}_0$ is the parameter to be determined.

If (15) is put into the differential equation, every d/dt just gives a factor of $(i\omega)$, so from (14), we obtain

$$\tilde{D}(\omega)\,\tilde{x}_0 \quad = \quad \tilde{f}_0 \tag{16}$$

where $\tilde{D}$ is the polynomial defined in (5). In the above, a common factor of $e^{i\omega t}$ has been cancelled — and this is the reason for assuming a solution of the form (15), so that every term has the same time dependence. In fact, this is also why we need to go to the complex case: a real solution say $\cos\omega t$ would not go back to the same time dependence after an odd number of differentiations. The solution is trivial:

$$\begin{aligned} \tilde{x}_0 \quad &= \quad \frac{\tilde{f}_0}{\tilde{D}(\omega)} \equiv A\, e^{i\theta} \\ \tilde{x}(t) \quad &= \quad \frac{\tilde{f}_0}{\tilde{D}(\omega)}\, e^{i\omega t} = A\, e^{i(\omega t + \theta)} \\ x(t) \quad &= \quad \Re\,\tilde{x}(t) = A\,\cos(\omega t + \theta) \end{aligned} \tag{17}$$

- Note that ψ, unlike the phases ϕ_j in (8), are not free parameters to be determined by the initial condition, but is fixed by the parameters appearing in the ODE.

- In fact, there is no free parameters at all.

- In physical terms, the solution has the same periodicity as the inhomogeneity (the "driving force"), but is in general out of phase by an amount θ.

- In general, we have to add a homogeneous solution $x_h(t)$, and the necessary free parameters appear in $x_h(t)$.

- More generally, the inhomogeneous term may consist of a sum of harmonic terms with different frequencies. We simply apply superposition: use the above method for each term, and add up the result.

5.5 Impulsive inhomogeneity*

**This subsection is more advanced and can be skipped.*

Here we restrict to a second-order ODE with constant coefficients, say

$$\left(\frac{d^2}{dt^2} + \omega_0^2\right) x(t) \quad = \quad f(t) \tag{18}$$

which can be thought of as the equation of motion for a unit mass (or $f(t) = F(t)/m$ where the mass is m) subject to (a) the restoring force of a spring $-\omega_0^2 x$, and (b) an external driving force $f(t)$; an additional linear damping force can also be handled.

First consider a special case where $f(t)$ (which we shall refer to as a force) is an *impulse*, in the following sense (**Figure 2a**).

- It is nonzero only for a short interval Δt around a time t_1.

- During this short interval it has an extremely large value f_1.

- We imagine $\Delta t \to 0$, $f_1 \to \infty$, but the product $f_1 \Delta t$ being finite.

We divide the analysis into three domains.

Before the impulse
Before the impulse there is no displacement;

$$x(t) \quad = \quad 0 \qquad \text{for } t < t_1$$

This means that the overall solution must contain a factor of $\Theta(t-t_1)$, where the Θ-function is defined as

$$\Theta(t) \quad = \quad \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t > 0 \end{cases}$$

After the impulse
After the impulse has passed, $x(t)$ satisfies the homogeneous equation, so must take the form

$$x(t) \quad = \quad B \cos \omega_0 (t-t_1) + C \sin \omega_0 (t-t_1)$$

where B and C are constants to be determined from the initial conditions — "initial" means immediately after the impulse, denoted as $t = t_1^+$. The arguments in the two terms have been chosen to refer to $t-t_1$, for convenience below.

In general (e.g., if there is damping), we would still have two coefficients multiplying two homogeneous solutions.

During the impulse
During the short interval Δt, there are two forces (or two terms in the differential equation in addition to d^2x/dt^2): (a) the restoring force of magnitude $\omega_0^2 x$, which is finite, and (b) the external force $f_1 \to \infty$. Therefore the former is neglected. Thus we have a case of uniform force and uniform acceleration, and at the end of the impulse we have

$$\begin{aligned} \dot{x}(t_1^+) \quad &= \quad f_1 \Delta t \\ x(t_1^+) \quad &= \quad \frac{1}{2} f_1 (\Delta t)^2 \end{aligned}$$

with the latter being zero in the limit under consideration.

Matching across the impulse
Thus we have $B = 0$ and $C\omega_0 = f_1 \Delta t$, giving

$$x(t) \quad = \quad \frac{f_1 \Delta t}{\omega_0} \sin \omega_0 (t-t_1)\, \Theta(t-t_1) \tag{19}$$

the Θ function having been inserted to make this valid for all t.

General force
A general force $f(t)$ can be regarded as the sum of impulses (**Figure 2b**), with magnitude $f_j = f(t_j)$ at the time t_j. Then, by superposition, the solution in this case is

$$
\begin{aligned}
x(t) &= \sum_j \frac{f(t_j)\Delta t}{\omega_0} \sin\omega_0(t-t_j)\,\Theta(t-t_j) \\
&= \int f(t')\,\frac{\sin\omega_0(t-t')}{\omega_0}\;\Theta(t-t')\,dt' \\
&= \int G(t-t')\,f(t')\,dt' \qquad (20)\\
G(t) &= \frac{1}{\omega_0}\,\sin\omega_0 t\,\Theta(t) \qquad (21)
\end{aligned}
$$

The formula (20) in principle gives the particular solution for *any* inhomogeneity, now not necessarily an impulse — at least the problem is reduced to an integral. This trick is fairly standard for inhomogeneous equations — replace the inhomogeneity to a point "source" (a "point" in time would be an impulse) and then superpose. The solution G for a point source is referred to as the *Green's function.* This idea will be repeated when we come to forced oscillations in the presence of damping.

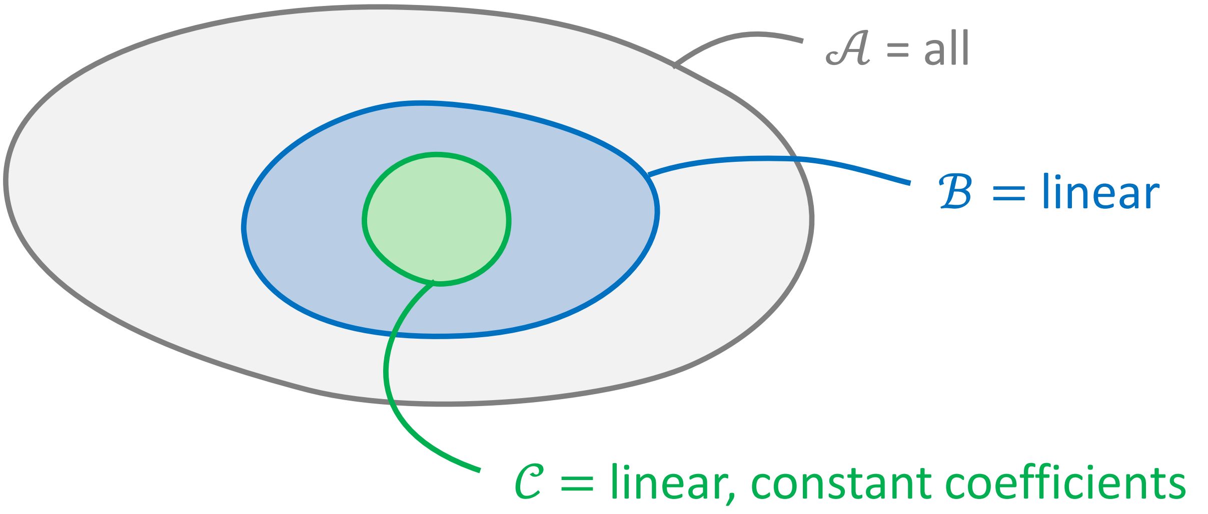

Figure 1

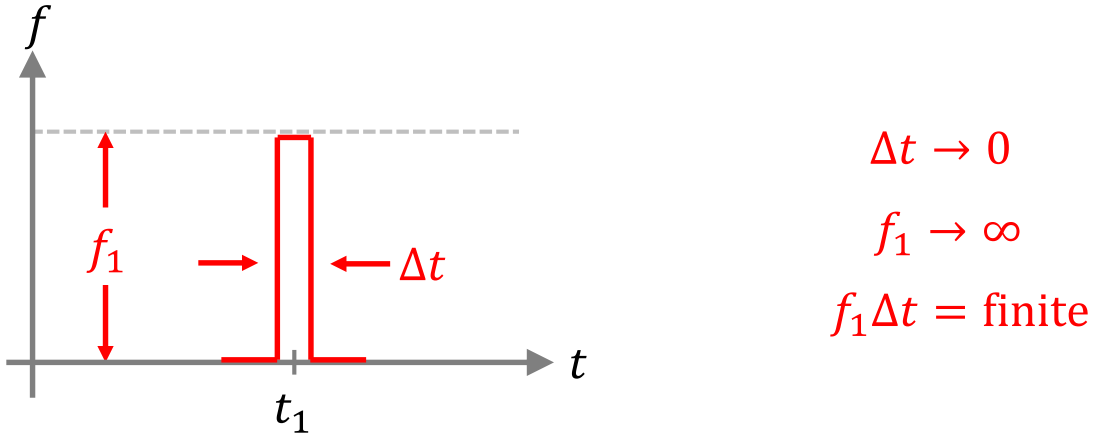

Figure 2a

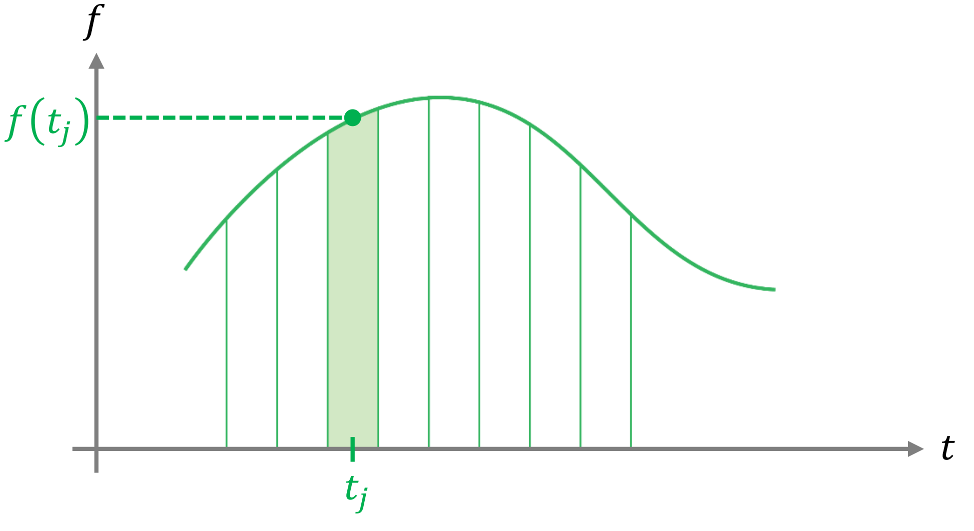

Figure 2b