Rotation: Part 2

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A more advanced treatment of rotational dynamics is given, using a vector approach.

Contents

1 Introduction

The state of a rigid body is specified by (a) the linear displacement $\vec{r}_C$ of a point C on the body (e.g., its CM), and (b) the *orientation* of the body about C (**Figure 1**). The latter can be imagined by pinning the body at C, and turning it in all possible ways.

Some feature of rotations[1] can be illustrated by the example of a cube (**Figure 2**). The three faces that are not visible are colored in a lighter version of the respective opposite faces. For simplicity consider only rotations by 90° about the x, y and z axes, denoted respectively as R_x, R_y, R_z.

To deal with rotations about an *arbitrary axis*, or equivalently about a *point*, requires generalization beyond the simpler formalism developed for rotations about a *fixed axis*.

1.1 Right hand rule

The *sign convention* goes as follows. When we speak of a rotation by some angle ϕ about an axis $\hat{\mathbf{n}}$, we point the thumb of the right hand along $\hat{\mathbf{n}}$, and the other fingers point towards positive ϕ. Thus rotations by $+90°$ along $+x$ and $+90°$ along $+y$ are shown in **Figure 3a** and **Figure 3b**. Of course ϕ can be any value; we use 90° rotations only for convenience.

1.2 Specifying a rotation

Unit vector and an angle
A rotation can therefore be specified by a unit vector $\hat{\mathbf{n}}$ (the axis of rotation) and the angle of rotation ϕ. Thus for **Figure 3a**, $\hat{\mathbf{n}} = \hat{\mathbf{i}}$ and $\phi = \pi/2$. Since

[1] The term "rotation" can mean the rotated configuration, or to the *change* of this configuration with time. The meaning should be clear from the context.

a magnitude and a direction are required, it will be *natural to adopt a vector language*; see (11) below.

Three angles

Any vector can be described by polar coordinates: a length and two angles. For a unit vector, the length is trivial, so $\hat{\mathbf{n}}$ is specified by a polar angle θ and an azimuthal angle ψ.[2] In fact

$$\begin{aligned} \hat{\mathbf{n}}_x &= \sin\theta\cos\psi \\ \hat{\mathbf{n}}_y &= \sin\theta\sin\psi \\ \hat{\mathbf{n}}_z &= \cos\theta \end{aligned} \tag{1}$$

Thus, the orientation of a rigid body can also be specified by three angles: θ, ψ, ϕ.[3]

1.3 Non-commutativity*

**This advanced topic is mentioned only in passing, and should be skipped in the first round of study.*

Translations

To appreciate the peculiarity of rotations, it is useful to compare with linear translations. **Figure 4** shows four states or configurations of an object in the x–y plane, with the object placed at the corners of a unit square, and labelled as C_0, C_1, C_2, C_3. Let T_x and T_y be the operation of translation (or shifting) the object by 1 unit in the x and y directions respectively. Then in obvious notation

$$\begin{aligned} &C_0 \xrightarrow{T_x} C_1 \\ &C_0 \xrightarrow{T_y} C_3 \end{aligned} \tag{2}$$

etc.

Now let us combine the two operations, but in different orders:

$$\begin{aligned} &C_0 \xrightarrow{T_x} C_1 \xrightarrow{T_y} C_2 \\ &C_0 \xrightarrow{T_y} C_3 \xrightarrow{T_x} C_2 \end{aligned} \tag{3}$$

The result is the same in both cases, as illustrated in **Figure 5**. The order does not matter.

It is common to express the same idea in a slightly different notation. Denote the configurations or states as $|C_0\rangle$ etc., and the relationships in (2) as

$$\begin{aligned} T_x\,|C_0\rangle &= |C_1\rangle \\ T_y\,|C_0\rangle &= |C_3\rangle \end{aligned} \tag{4}$$

which should be read as the *operators* T_x, T_y acting on $|C_0\rangle$ giving the states on the RHS. The composite operations in (3) would be expressed as

$$\begin{aligned} T_y\,T_x\,|C_0\rangle &= |C_2\rangle \\ T_x\,T_y\,|C_0\rangle &= |C_2\rangle \end{aligned} \tag{5}$$

The product of operators should be read from right to left; for example, in the first line above, T_x acts first, and then T_y acts on the result. In short we have

$$T_y\,T_x\,|C_0\rangle = T_x\,T_y\,|C_0\rangle \tag{6}$$

But this is true not just for the state $|C_0\rangle$, but for any state, so we can state an operator relationship:

$$T_y\,T_x = T_x\,T_y \tag{7}$$

We say translations are *commutative*. All this abstract notation is just a fancy (and at some level unnecessarily fancy) way of expressing the idea in **Figure 5**.

Rotations

Some possible configurations of the cube are shown in **Figure 6**; the labelling of C_i is arbitrary. Rotations by R_x and R_y are shown in **Figure 7**; the results of the two different orders are different. Adopting an obviously parallel notation, we have

$$\begin{aligned} &C_0 \xrightarrow{R_x} C_1 \xrightarrow{R_y} C_2 \\ &C_0 \xrightarrow{R_y} C_3 \xrightarrow{R_x} C_4 \end{aligned} \tag{8}$$

or using a notation similar to (5)

$$\begin{aligned} R_y\,R_x\,|C_0\rangle &= |C_2\rangle \\ R_x\,R_y\,|C_0\rangle &= |C_4\rangle \end{aligned} \tag{9}$$

As an operator equation

$$R_y\,R_x \neq R_x\,R_y \tag{10}$$

Thus rotations are *not commutative.*

Therefore the analysis of rotations more complicated — and in the end more interesting and rewarding; but that is well beyond the level needed here. Fortunately, angular velocity involves only one infinitesimal rotation in an infinitesimal time. So there is no difficulty.

[2] Usually we would denote the azimuthal angle as ϕ, but now we reserve this symbol for the amount of rotation about $\hat{\mathbf{n}}$.

[3] These are not the same as the three Euler angles that more advanced students may have encountered, though the ideas are somewhat similar.

1.4 Reducing to special case

If the rotation is constrained to be about a fixed axis (say the z-axis), then there is only one angle ϕ and rotations are commutative. This case has been discussed before, without the need for vector notation. It is important to check that every formula in the present module agrees with the earlier discussion in this special case. We shall not explicitly carry out the check in every instance.

2 Angular displacement and velocity

2.1 Vector form of infinitesimal rotation

With the above in mind, consider an infinitesimal rotation by an angle $\Delta\phi$ about an axis $\hat{\mathbf{n}}$. We denote such a rotation as a vector:[4]

$$\Delta\vec{\phi} \quad = \quad \Delta\phi\,\hat{\mathbf{n}} \tag{11}$$

namely, the direction giving the axis of rotation, and the magnitude giving the angle of rotation. Note that this definition is used only for an *infinitesimal* rotation. Henceforth, we do not deal with finite rotations.

2.2 Displacement

Consider a point P at a position $\vec{r}$ from the origin O on the axis. The perpendicular distance from the axis is $r_\perp$ to P (**Figure 8a**). Under a rotation by an infinitesimal angle $\Delta\phi$, P is displaced by $\Delta\vec{s}$.[5] The magnitude of displacement is $\Delta s = \Delta\phi \cdot r_\perp = \Delta\phi \cdot r \cdot \sin\Theta$, where Θ is the angle between $\hat{\mathbf{n}}$ and $\vec{r}$. The direction is perpendicular to both $\hat{\mathbf{n}}$ and $\vec{r}$ (i.e., into the page in **Figure 8a**). The two properties are summarized in one vector statement:

$$\boxed{\Delta\vec{s} = \Delta\vec{\phi} \times \vec{r}} \tag{12}$$

[4]Strictly speaking, the vector sign should be placed on the whole symbol $\Delta\phi$.

[5]Likewise, the vector sign should strictly speaking be placed on the whole symbols Δs.

2.3 Angular velocity

Upon dividing by the time Δt, we get

$$\boxed{\vec{v} = \vec{\omega} \times \vec{r}} \tag{13}$$

where $\vec{v} = \Delta\vec{s}/\Delta t$ is the linear velocity of P and $\vec{\omega}$ is the angular velocity, defined as a vector:

$$\vec{\omega} \quad = \quad \frac{\Delta\vec{\phi}}{\Delta t} \tag{14}$$

Its magnitude agrees with the familiar definition, and it is given a direction along the axis of rotation $\hat{\mathbf{n}}$.

The formula (13) deals with finite quantities and is more convenient compared with (12), which deals with infinitesimal quantities.

The vector angular acceleration

$$\vec{\alpha} \quad = \quad \frac{d\vec{\omega}}{dt} \tag{15}$$

turns out to be less useful; see Section 4.2.

3 Torque

3.1 Single force

As before, torque is defined as the work done per unit angular displacement. Consider a force $\vec{F}$ acting on a point mass P at a point $\vec{r}$ measured from the origin O, and causing an infinitesimal displacement that can be regarded as a rotation (**Figure 9a**). The work done is, in obvious notation,

$$\begin{aligned} & \Delta W = \vec{F} \cdot \Delta\vec{s} \\ = \quad & \vec{F} \cdot \left(\Delta\vec{\phi} \times \vec{r}\right) = \Delta\vec{\phi} \cdot \left(\vec{r} \times \vec{F}\right) \\ \equiv \quad & \vec{\tau} \cdot \Delta\vec{\phi} \end{aligned} \tag{16}$$

where we have used the identity (see Appendix A)

$$\vec{a} \cdot (\vec{b} \times \vec{c}) \quad = \quad \vec{b} \cdot (\vec{c} \times \vec{a}) \tag{17}$$

and in the last line of the derivation we have defined the torque

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}} \tag{18}$$

In regarding the displacement as an infinitesimal rotation, we are implicitly assuming a rigid body.

But that assumption merely *motivates* the definition (18), and the various consequences below depend only on the definition — and are valid even for a system of particles that do not constitute a rigid body.

A special case is shown in **Figure 9b**, for $\vec{r}$ and $\vec{F}$ both in the x–y plane. It is clear that $\vec{\tau}$ is perpendicular to the plane, with magnitude $\tau = r_\perp F = rF_\perp$, in agreement with the definition adopted earlier.

3.2 Newton's third law for torques

Case of linear motion
Consider forces F_1 and F_2 acting on a system of two masses (**Figure 10a**). Is the total force F given simply by $F_1 + F_2$? Should there also be the forces P and Q (**Figure 10b**), which are the force on 1 due to 2, and the force on 2 due to 1? Should we not have $F = F_1 + F_2 + P + Q$? The answer is simple: It does not matter; $P + Q = 0$ because of Newton's third law. Thus total force on a system is the same as total *external* force.

Case of rotational motion
Now consider the case of torques. **Figure 11** shows two particles 1 and 2, with internal forces $\vec{P}$ and $\vec{Q}$ forming an action-reaction pair. There is an internal torque on 1 due to $\vec{P}$:

$$\vec{\tau}_1^{\,\mathrm{int}} = \vec{r}_1 \times \vec{P} \tag{19}$$

Likewise there is an internal torque on 2 due to $\vec{Q}$:

$$\vec{\tau}_2^{\,\mathrm{int}} = \vec{r}_2 \times \vec{Q} = -\vec{r}_2 \times \vec{P} \tag{20}$$

where we have used Newton's third law for the two forces. The sum of the two torques is

$$\vec{\tau}_1^{\,\mathrm{int}} + \vec{\tau}_2^{\,\mathrm{int}} = \vec{r}_{12} \times \vec{P} \tag{21}$$

where $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2$ is the vector joining the two particles. But $\vec{P}$ must point along this line, so the cross product is zero. Thus we prove the analog of Newton's third law: the total torque is the same as the total *external* torque.

4 Equation of motion

4.1 Derivation

Imagine external forces $\vec{F}_\alpha$ acting on a system of particles α at positions $\vec{r}_\alpha$. Then

$$\vec{\tau} = \sum_\alpha \vec{\tau}_\alpha = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha \tag{22}$$

Because of the result in the last Section, we can replace $\vec{F}_\alpha$ by $\vec{F}'_\alpha$, the *total* force acting on α, i.e., adding back the internal forces (which will cancel in the total torque). But

$$\vec{F}'_\alpha = \frac{d\vec{p}_\alpha}{dt} \tag{23}$$

where $\vec{p}_\alpha$ is the momentum of particle α. Thus

$$\vec{\tau} = \sum_\alpha \vec{r}_\alpha \times \frac{d\vec{p}_\alpha}{dt} \tag{24}$$

Next every term in the sum can be replaced by

$$\frac{d}{dt}\left(\vec{r}_\alpha \times \vec{p}_\alpha\right) \tag{25}$$

since the extra term

$$\frac{d\vec{r}_\alpha}{dt} \times \vec{p}_\alpha = \vec{v}_a \times \vec{p}_a = 0 \tag{26}$$

because $\vec{v}$ and $\vec{p}$ are parallel.

We then obtain, taking the derivative outside the sum

$$\vec{\tau} = \frac{d}{dt} \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha \tag{27}$$

We are then led to define the *angular momentum* of a particle as

$$\vec{L}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha \tag{28}$$

and the total angular momentum as

$$\vec{L} = \sum_a \vec{L}_\alpha \tag{29}$$

while the equation of motion is

$$\vec{\tau} = \frac{d\vec{L}}{dt} \tag{30}$$

4.2 Applies to non-rigid bodies

The above derivation did not assume that the system is a rigid body. In an earlier module, we had encountered the question as to whether

$$\tau = \frac{dL}{dt} = \frac{d(I\omega)}{dt} \qquad ?? \qquad (31)$$

$$\tau = I\frac{d\omega}{dt} = I\,\alpha \qquad ?? \qquad (32)$$

The two versions differ if the body is not rigid and I is not constant. We had previously asserted, without proof, that (31) is correct but (32) is not. Now we have provided the proof.

4.3 Precession of a top

From the above, a key concept in rotational dynamics is the rate of change of $\vec{L}$. There are two categories of effects (and of course their combinations).

- If rotation occurs about a fixed axis, only the magnitude $L = |\vec{L}|$ changes. These situations do not need a vector approach, and have been discussed at length earlier.
- In other situations, an external torque may causes the *direction* of $\vec{L}$ to change with time.

The latter situation can be illustrated by the precession of a top. A top supported on the ground at O is spinning rapidly about its symmetry axis OA, at angular velocity ω; its CM is at a displacement $\vec{R}$ from O, on the axis OA. At a certain moment, the axis OA lies in the x–z plane (z-axis vertically upwards), tilted at an angle θ from the vertical (**Figure 12a**). The top will *precess* slowly, i.e., the point A will describe a circle in the x–y plane, as shown by the dotted line. We want to understand the reason for the precession, and to determine its rate.

The angular momentum of the spinning top has a magnitude $L = I\omega$, where I is the moment of inertia about the axis OA. The direction of $\vec{L}$ is along OA, so

$$\vec{L} = I\omega\,(\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{k}}) \qquad (33)$$

There is a force acting on the top, due to gravity

$$\vec{F} = -\,Mg\,\hat{\mathbf{k}} \qquad (34)$$

which leads to a torque about O given by

$$\vec{\tau} = \vec{R}\times\vec{F} \qquad (35)$$

where

$$\vec{R} = R\,(\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{k}}) \qquad (36)$$

Thus

$$\begin{aligned}\vec{\tau} &= MgR\,[(\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{k}})\times(-\,\hat{\mathbf{k}})] \\ &= MgR\,\sin\theta\,\hat{\mathbf{j}} \qquad (37)\end{aligned}$$

So, after a short time Δt, $\vec{L}$ will gain a y-component given by

$$L_y = \tau_y\Delta t = MgR\sin\theta\,\Delta t \qquad (38)$$

while L_x is to first order unchanged. In **Figure 12b**, the arrow indicates the projection of $\vec{L}$ onto the x–y plane. The angle $\Delta\psi$ is

$$\Delta\psi = \frac{L_y}{L_x} = \frac{MgR\,\sin\theta}{I\omega\sin\,\theta} \qquad (39)$$

Hence ψ increases at a rate

$$\Omega \equiv \frac{\Delta\psi}{\Delta t} = \frac{MgR}{I\omega} \qquad (40)$$

This is the angular precession frequency of the top. If ω is large, as assumed, then Ω is small. The angular momentum associated with the precession itself has been neglected in the above analysis.

5 Angular momentum

5.1 Definition

For a single particle, and now for convenience dropping the label α, the angular momentum is

$$\boxed{\vec{L} = \vec{r}\times\vec{p}} \qquad (41)$$

Referring to **Figure 13**, we see that $\vec{L}$ is perpendicular to the plane containing $\vec{r}$ and $\vec{p}$, and has a magnitude $L = r_\perp p = rp_\perp = rp\sin\Theta$, where Θ is the angle between the two vectors.

5.2 Conservation of angular momentum

General statement
Since $\vec{\tau} = d\vec{L}/dt$, angular momentum is conserved if the torque is zero.

Central force
The most important application concerns a particle (e.g., a planet) subject to a central force (e.g., gravitational force due to the sun). If the force is central, then $\vec{F}$ acting on the particle is parallel to $\vec{r}$, and $\vec{\tau} = \vec{r} \times \vec{F} = 0$. So angular momentum is conserved for a particle subject to a central force.

Planar planetary orbits
First of all, the *direction* of $\vec{L}$ is unchanged. Choose the z-axis along $\vec{L}$. Then $\vec{r}$ and $\vec{p}$ are in the x–y plane. In other words, planetary orbits stay in one plane; motion such as that sketched in **Figure 14** is not allowed.

Kepler's second law
Consider planetary motion over a short time interval Δt (**Figure 15**). The sun is at the origin O and the planet moves by an amount $\Delta\vec{s} = \vec{v}\Delta t$. The area ΔA swept out by the radius vector is the shaded triangle. (The "extra" small triangle has a negligible area $\propto (\Delta s)^2 \propto (\Delta t)^2$.)

$$\begin{aligned} \Delta A &= \frac{1}{2} r\,\Delta s_\perp = \frac{1}{2} r\,v_\perp\,\Delta t \\ \frac{\Delta A}{\Delta t} &= \frac{1}{2} r\,v_\perp = \frac{1}{2m} r\,p_\perp = \frac{L}{2m} \end{aligned} \tag{42}$$

where m is the mass of the planet and L its angular momentum.

Since the force is central, L is conserved, and we conclude that the radius vector sweeps out equal areas in equal times — Kepler's second law. Historically, the chain of deduction was the reverse: Kepler noticed this regularity from astronomical observations, and Newton used this empirical law to deduce that gravity was a central force.

A consequence is that planets move slower when farther from the sun (**Figure 16**).

Atomic physics
Consider the motion of the electron around the nucleus in a hydrogen atom. The force is again central (in fact, also inverse-square, but that is not relevant here). In this case, there are two important properties.

- The value of the angular momentum L is constant. We can therefore label the motion (or "state") of the electron by the constant value of L. This fact follows from the above discussion.
- The value of L can only be an integral multiple of a basic unit, namely $\hbar$: $L = \ell\,\hbar$, where $\ell = 0, 1, 2, \ldots$. This fact comes from quantum mechanics. (Actually, to be more precise, the statement is that $L^2 = \ell(\ell{+}1)\,\hbar^2$.)

The state of the electron with $\ell = 0, 1, 2, \ldots$ are called the $s, p, d, \ldots$ orbitals. Their properties are central to chemical bonding.

6 Moment of inertia

6.1 An example

Description of example
This Section deals with the moment of inertia in a more general context. Heuristically, I is the proportionality constant between L and ω:

$$L \sim I\,\omega \tag{43}$$

Now we realize that $\vec{L}$ and $\vec{\omega}$ are vectors. But are they always in the same direction? If not, do we need to generalize the concept of I? We first introduce these ideas through a simple example.

A dumbbell consists of two masses, each m, at the ends of a light rod of length $2R$. The dumbbell is attached to an axle (chosen as z-axis) through its CM, and is inclined at an angle θ to the axle (**Figure 17a**). The axle is rotated at an angular velocity ω. Thus,

$$\vec{\omega} = \omega\,\hat{\mathbf{k}} \tag{44}$$

We now want to find the angular momentum.

Take the point mass at $z > 0$. Its position vector $\vec{r}$ (on the page) and its momentum $\vec{p}$ (into the page) are perpendicular, so its its angular momentum $\vec{L}_1 = \vec{r} \times \vec{p}$ has magnitude $Rp = R \cdot m(R\omega\,\sin\theta) = mR^2\omega\,\sin\theta$. The direction of $\vec{L}_1$ is shown by the red arrow in **Figure 17a**. Adding the other point mass just doubles the angular momentum (since both $\vec{r}$ and $\vec{p}$ are reversed). Thus

$$L = 2mR^2\omega\sin\theta \tag{45}$$

with components

$$L_x(0) = -L\cos\theta$$
$$L_z(0) = L\sin\theta \qquad (46)$$

Problem 1
Go back to the first particle and express $\vec{r}$ and $\vec{p} = m\vec{v} = m\vec{\omega}\times\vec{r}$ in Cartesian coordinates and hence calculate the Cartesian components of $\vec{L}_1$ and $\vec{L}$. Verify the answer above. §

Two vectors not parallel
We draw a lesson from this example: $\vec{\omega}$ and $\vec{L}$ need not be in the same direction. Somehow, the proportionality "constant" in $L \sim I\,\omega$ cannot be just one number I.

Torque required
The angular momentum $\vec{L}$ is the red arrow attached rigidly, at right angles, to the dumbbell. As the system is turned on the axle, the dumbbell and the red arrow rotate. Thus the angular momentum changes with time, and a torque is required.

Problem 2
Refer to the top view of the situation in **Figure 17b**. The red arrow represents the projection of $\vec{L}$ onto the x–y plane, with initial value $L_x(0)$ given by (46). As time goes on

$$L_x(t) = L_x(0)\cos\omega t$$
$$L_y(t) = L_x(0)\sin\omega t \qquad (47)$$

Find the torque $\tau(t)$ for all times. §

In a sense, the details do not matter. We only emphasize two points.

- $\vec{L}$ and $\vec{\omega}$ need not be in the same direction.
- Although $\vec{\omega}$ is constant, $\vec{L}$ is not, and therefore $\vec{\tau} \neq 0$. This example again illustrates the fact that we cannot have an equation of the form $\tau = I\,(d\omega/dt)$.

6.2 Derivation of key formula*

** This part is more advanced and can be skipped.*

Consider a collection of particles α constituting a rigid body, and the total angular momentum

$$\vec{L} = \sum_\alpha \vec{r}^\alpha \times \vec{p}^\alpha \qquad (48)$$

To avoid confusion, for vector quantities, the index α labelling particles is written as a superscript, while the indices i, j labelling the Cartesian directions are written as subscripts.

Using the formulas derived, we have

$$\begin{aligned}\vec{L} &= \sum_\alpha \vec{r}^\alpha \times \vec{p}^\alpha \\ &= \sum_\alpha \vec{r}^\alpha \times (m_\alpha\,\vec{v}^\alpha) \\ &= \sum_\alpha m_\alpha\,[\vec{r}^\alpha \times (\vec{\omega}\times\vec{r}^\alpha)] \qquad (49)\end{aligned}$$

We have used (13) to express $\vec{v}^\alpha$ in terms of $\vec{\omega}$ — which is the same for all particles. If we can take $\vec{\omega}$ outside the summation, the other factor depend only on m_α and $\vec{r}^\alpha$, and can be identified as the moment of inertia I. But the tricky point is that we have to deal with the Cartesian indices.

Use the vector identity (see Appendix A)

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}\,(\vec{a}\cdot\vec{c}) - \vec{c}\,(\vec{a}\cdot\vec{b}) \qquad (50)$$

to write (49) as

$$\vec{L} = \sum_\alpha m_\alpha\,[\vec{\omega}\,(\vec{r}^\alpha\cdot\vec{r}^\alpha) - \vec{r}^\alpha\,(\vec{\omega}\cdot\vec{r}^\alpha)] \qquad (51)$$

Take the i component on both sides:

$$L_i = \sum_\alpha m_\alpha[\omega_i (r^\alpha)^2 - r_i^\alpha \sum_j \omega_j r_j^\alpha] \qquad (52)$$

In the first term, write

$$\omega_i = \sum_j \delta_{ij}\,\omega_j \qquad (53)$$

Then on the RHS, both terms contain $\sum_j \ldots \omega_j$, and the other factors are

$$\boxed{I_{ij} \equiv \sum_\alpha m_\alpha\left[(r^\alpha)^2\,\delta_{ij} - r_i^\alpha r_j^\alpha\right]} \qquad (54)$$

which we define as the *moment of inertia*, and the angular momentum is given by

$$\boxed{L_i = \sum_j I_{ij}\,\omega_j} \qquad (55)$$

6.3 Properties of key formula

Matrix form
The key formula (55) shows that $\vec{L}$ and $\vec{\omega}$ are linearly related. But they are not necessarily in the same direction.

We can express (55) in terms of matrix multiplication, with $\sum_j$ being naturally implied:

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \quad (56)$$

or in an obvious shorthand

$$[L] = [I]\,[\omega] \quad (57)$$

with square brackets denoting appropriate column or square matrices.

A linear relationship between two vectors can always be expressed in such a matrix form. In short, the natural proportionality "constant" between two vectors is a matrix, i.e., an object with two indices.

Incidentally, to simplify notation, we shall henceforth (whenever there is no danger of confusion) drop the index α and write

$$\boxed{I_{ij} = \sum m\,(r^2\,\delta_{ij} - r_i r_j)} \quad (58)$$

it being understood that we sum over all masses.

Symmetry
It is obvious that $[I]$ is a symmetric matrix. (This is not necessarily true of all such proportionality matrices between two vectors.)

Diagonal components
Consider for example I_{33}. If (58) is evaluated for $i = j = 3$, we get $\delta_{ij} = 1$, so

$$\begin{aligned} I_{33} &= \sum m(r^2 - r_3^2) \\ &= \sum m(r_1^2 + r_2^2) = \sum m r_\perp^2 \end{aligned} \quad (59)$$

where $r_\perp$ is the perpendicular distance to the 3-axis. This recovers the familiar formula for the moment of inertia about the 3-axis.

Off-diagonal components
Consider for example I_{12}. If (58) is evaluated for $i = 1$, $j = 2$, we get $\delta_{ij} = 0$, so

$$I_{12} = -\sum m r_1 r_2 \quad (60)$$

If the object is symmetric under $r_1 \mapsto -r_1$ (e.g., the object in **Figure 18a**), then the RHS as an odd power of r_1 will sum to zero; likewise if the object is symmetric under $r_2 \mapsto -r_2$ (e.g., the object in **Figure 18b**). So for symmetrical object, $[I]$ is diagonal, in which case

$$\begin{aligned} L_1 &= I_{11}\,\omega_1 \\ L_2 &= I_{22}\,\omega_2 \\ L_3 &= I_{33}\,\omega_3 \end{aligned} \quad (61)$$

which combined with formulas such as (59) return us to the simpler situations discussed in *Rotation: Part 1*.

Problem 3
A dumbbell consists of two masses, each m, on the ends of a light rod of length $2R$. Find I_{ij} if the axes are chosen as in the three cases shown in **Figure 19**. Check the special cases $\theta = 0, \pi/2$. §

Tensor property
It is often said that I_{ij} is a *tensor* of rank 2. This carries two levels of meaning. (a) First, it means that I_{ij} has two indices. (b) Secondly, it means that there is a precise rule about how it transforms under a change of coordinates (in this case a rotation). The transformation property is such that if (57) holds in one system of coordinates $[L] = [I]\,[\omega]$, then it is guaranteed to hold in another system of coordinates $[L'] = [I']\,[\omega']$. We can bypass all such transformations by always calculating the new I'_{ij} afresh from the new position vectors $\vec{r}'$.

Appendix

A Vector identities

Theorem 1
The quantity $V(abc) = \vec{a} \cdot (\vec{b} \times \vec{c})$ is the same under cyclic permutation of the three vectors, i.e.,

$$V(abc) = V(bca) \quad (62)$$

Proof
Let $\vec{d} = \vec{b} \times \vec{c}$. Then

$$\begin{aligned} V(abc) &= \sum_i a_i d_i = \sum_i a_i \sum_{jk} \epsilon_{ijk} b_j c_k \\ &= \sum_{ijk} \epsilon_{ijk}\, a_i b_j c_k \end{aligned} \quad (63)$$

where ϵ_{ijk} is the totally antisymmetric symbol, e.g., $\epsilon_{123} = 1$, $\epsilon_{321} = -1$, and zero if any two indices are equal. The identity follows from the property that $\epsilon_{ijk} = \epsilon_{jki}$.

In fact, it is obvious that

$$V = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \quad (64)$$

and the identity follows from the property of a determinant under cyclic permutation of rows.

Problem 4
Consider a parallelepiped with sides $\vec{a}$, $\vec{b}$, $\vec{c}$. Show that the area of the base formed by the two sides $\vec{b}$ and $\vec{c}$ is given by $|\vec{b} \times \vec{c}|$ and the volume of the parallelepiped is exactly $|V(abc)|$. This gives a proof of the identity, at least up to a sign. §

Theorem 2
For any three vectors

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\,(\vec{a} \cdot \vec{b}) \quad (65)$$

Proof
Let the LHS be a vector $\vec{X}$ and let $\vec{d} = \vec{b} \times \vec{c}$.

$$\begin{aligned} X_i &= \sum_{jk} \epsilon_{ijk} a_j d_k \\ &= \sum_{jk} \epsilon_{ijk} a_j \sum_{mn} \epsilon_{kmn} b_m c_n \\ &= \sum_{jmn} \left(\sum_k \epsilon_{kij} \epsilon_{kmn} \right) a_j b_m c_n \end{aligned} \quad (66)$$

Problem 5
(a) By checking all cases, prove that

$$\sum_k \epsilon_{kij} \epsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \quad (67)$$

(b) Using (67), prove the identity (65). §

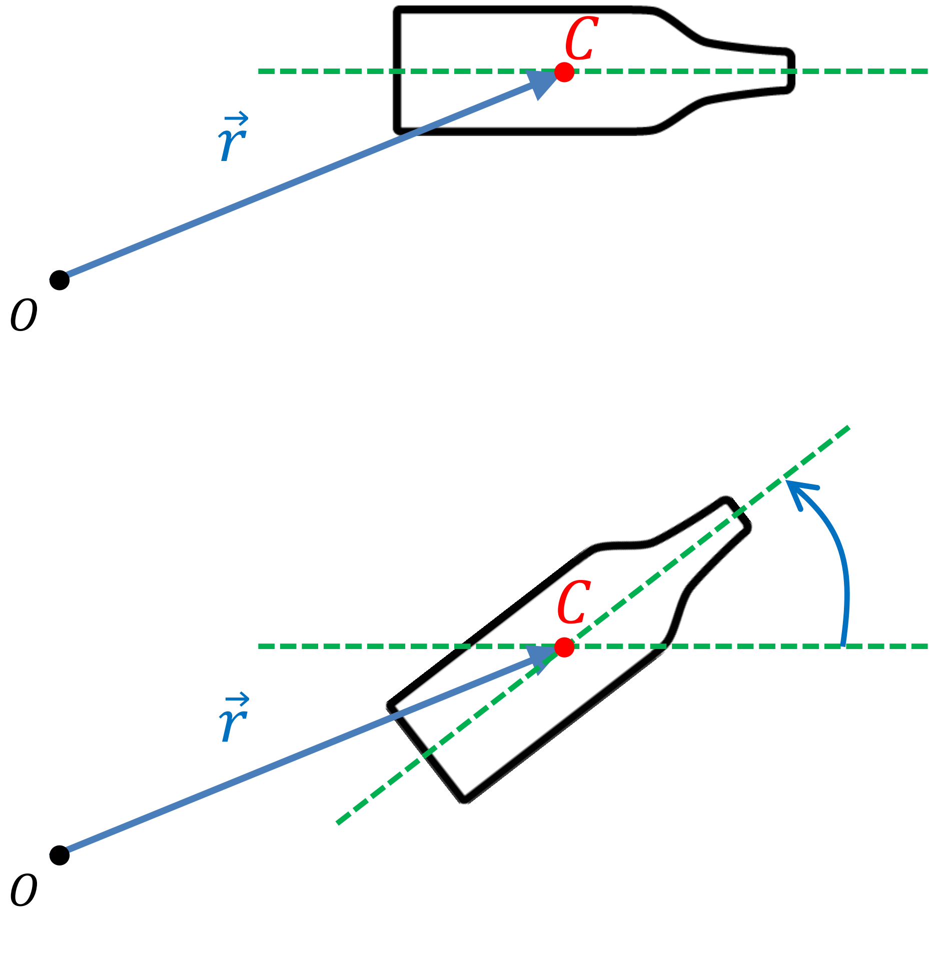


Figure 1

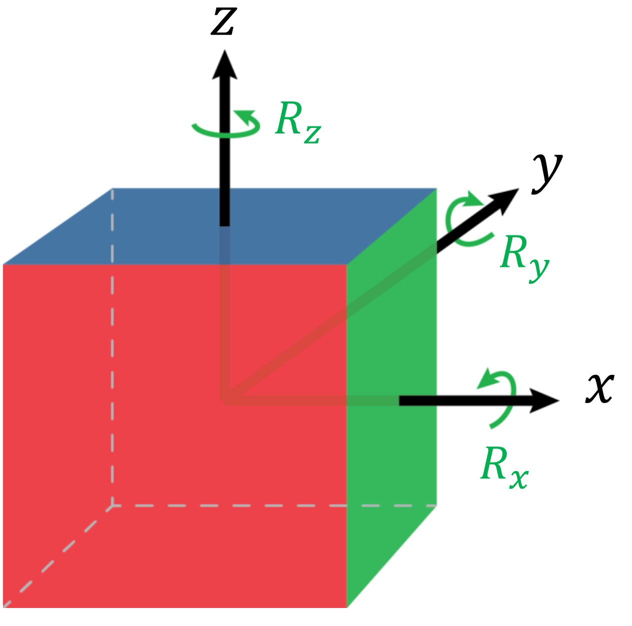


Figure 2

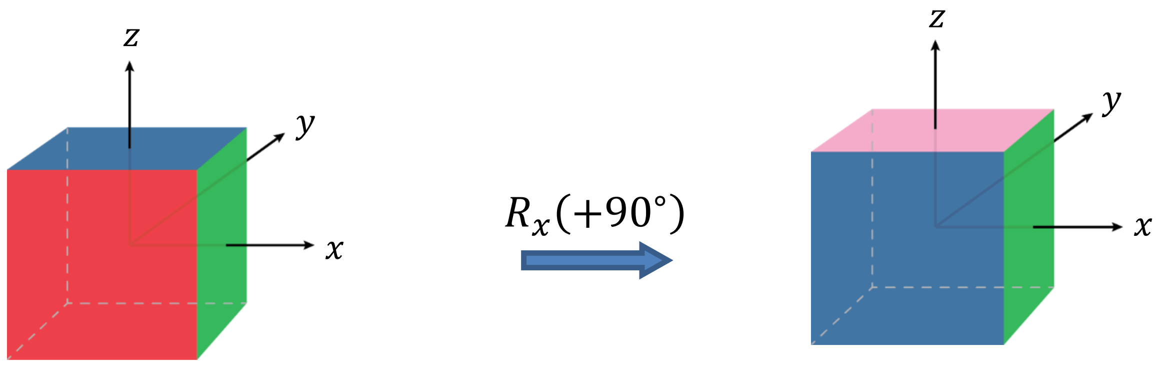


Figure 3a

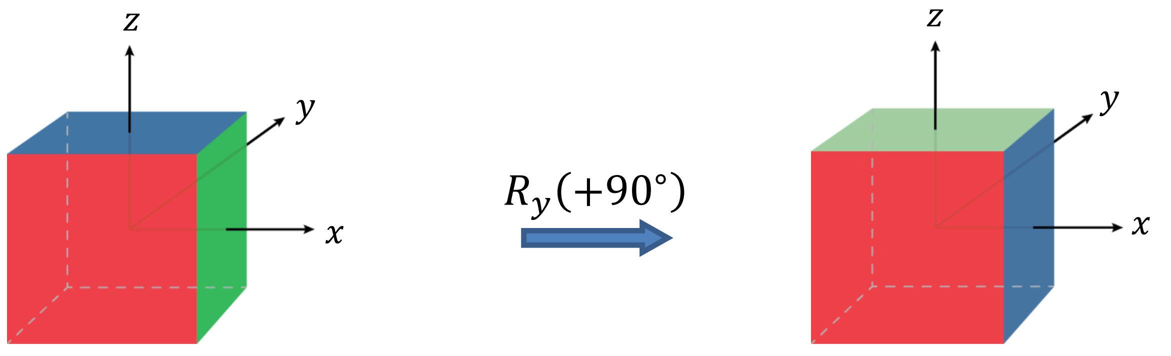


Figure 3b

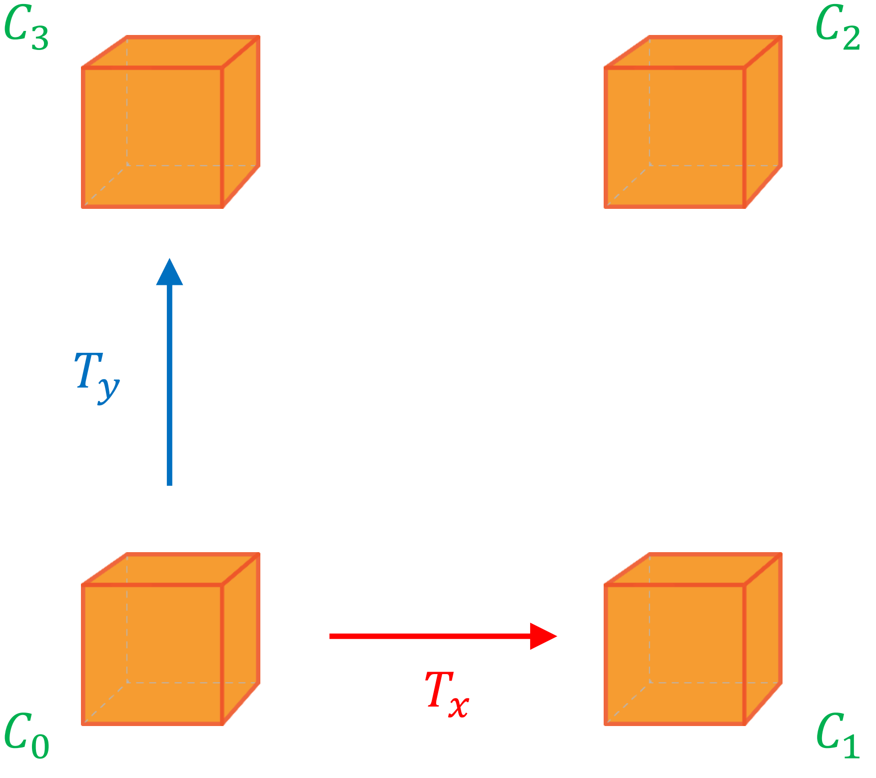


Figure 4

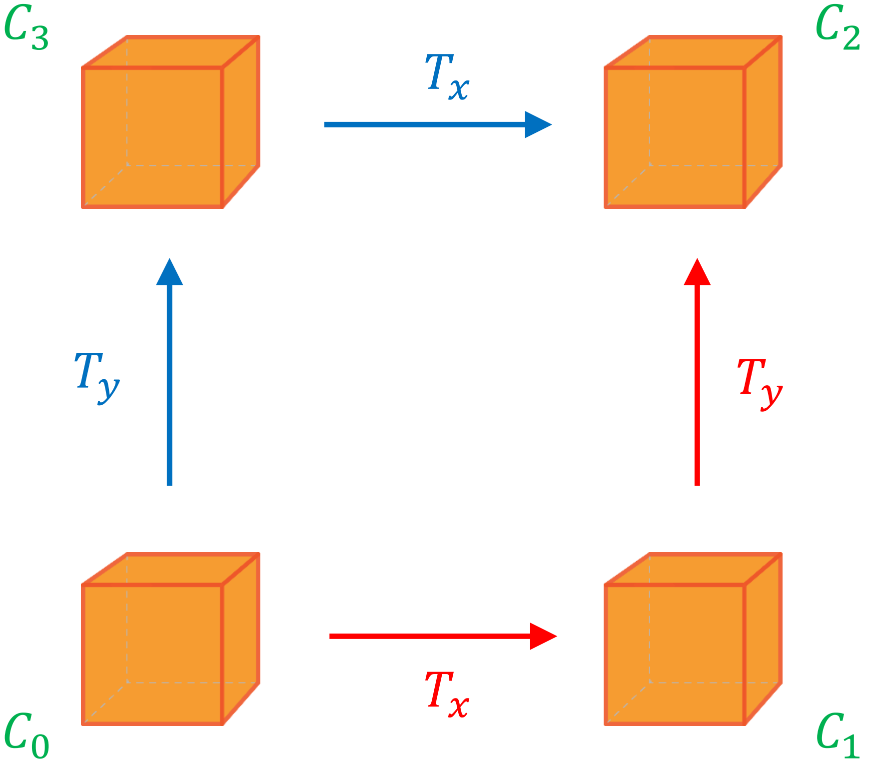


Figure 5

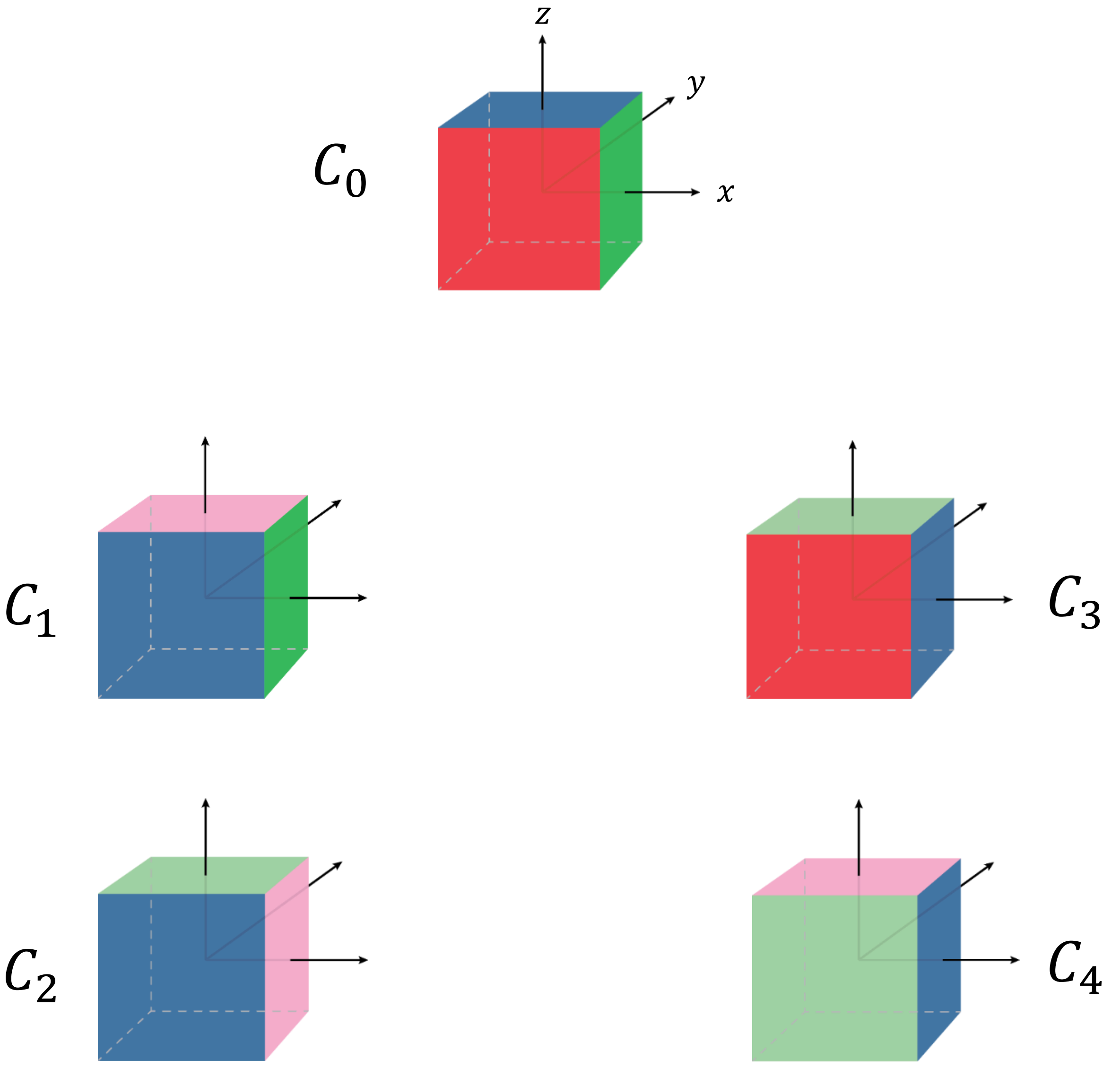
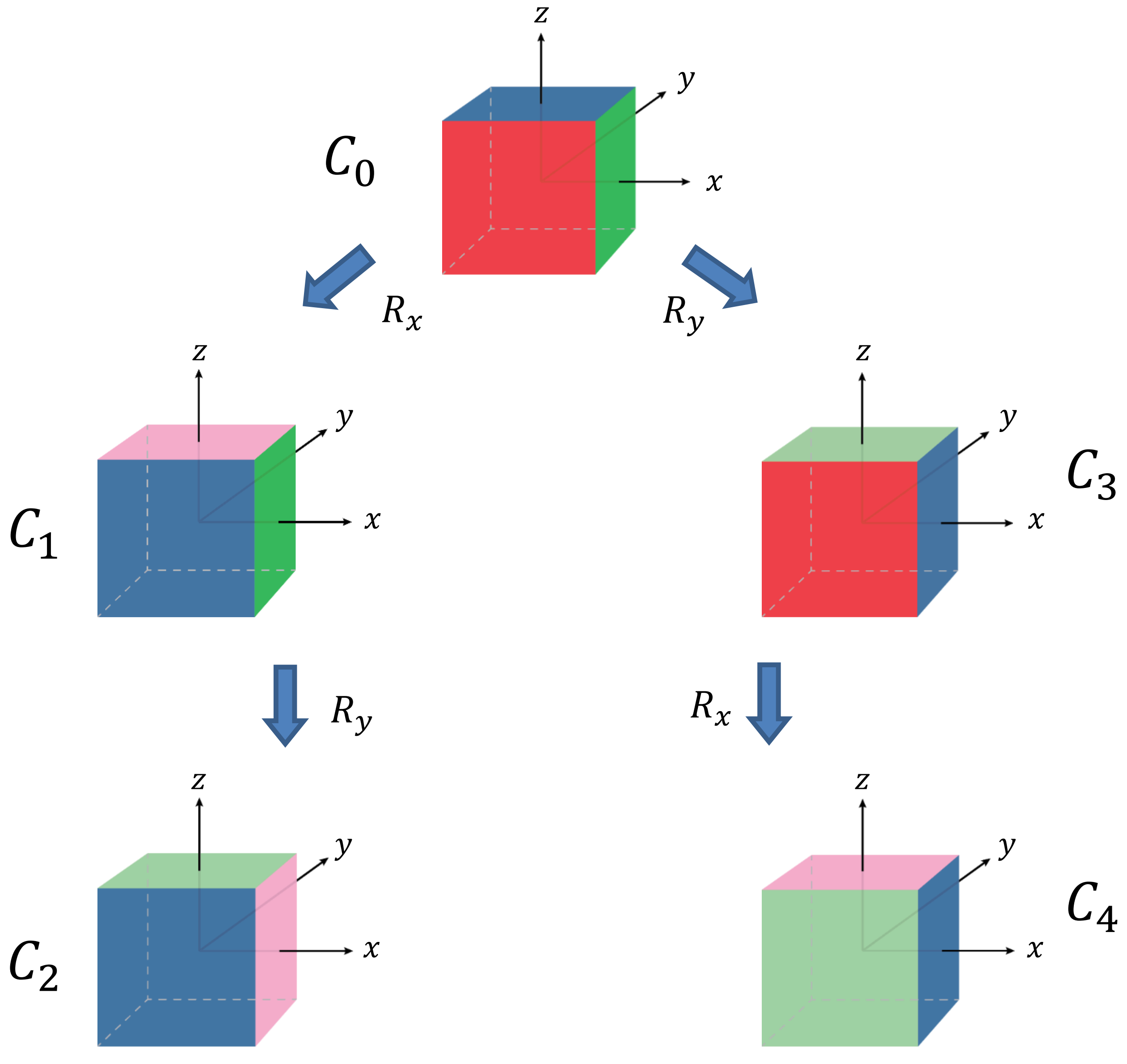


Figure 6



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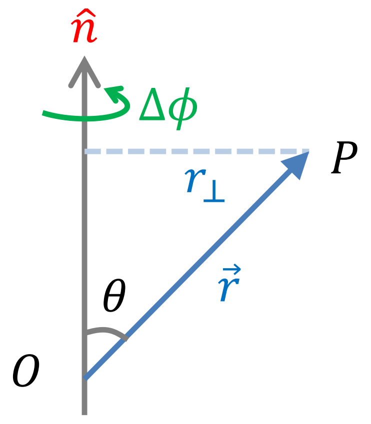
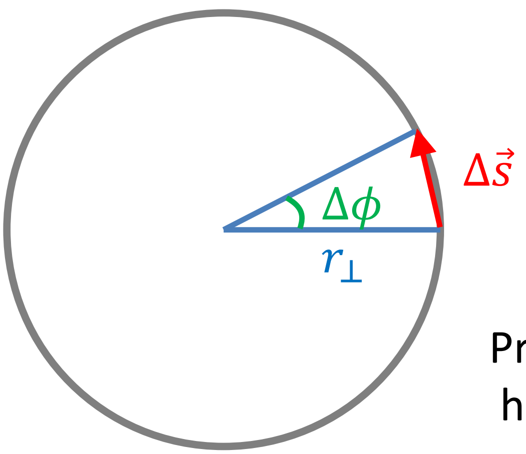


Figure 8a



Projection on the horizontal plane

Figure 8b

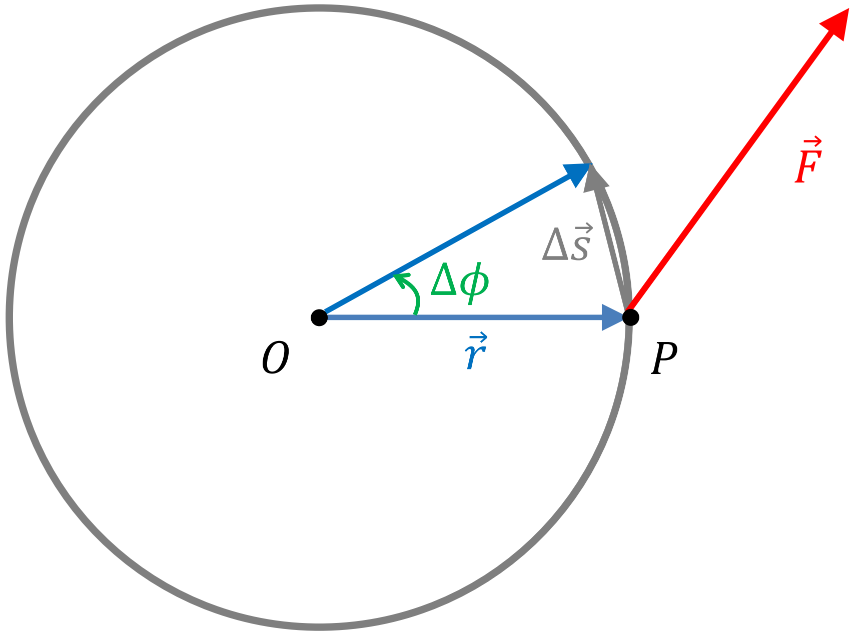
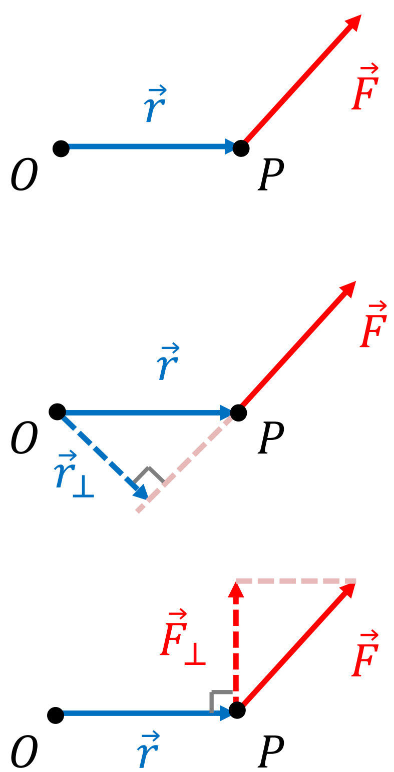


Figure 9a



$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = r_\perp \, F$$

$$\tau = r \, F_\perp$$

Figure 9b

Figure 10a

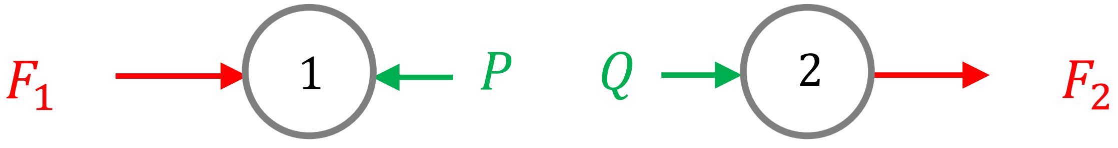


Figure 10b

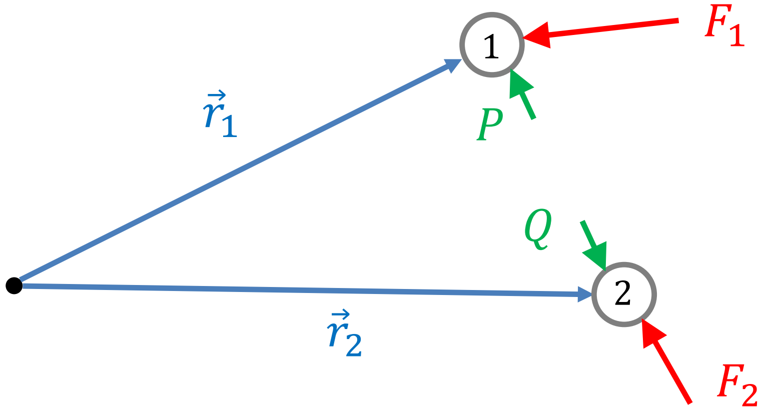


Figure 11

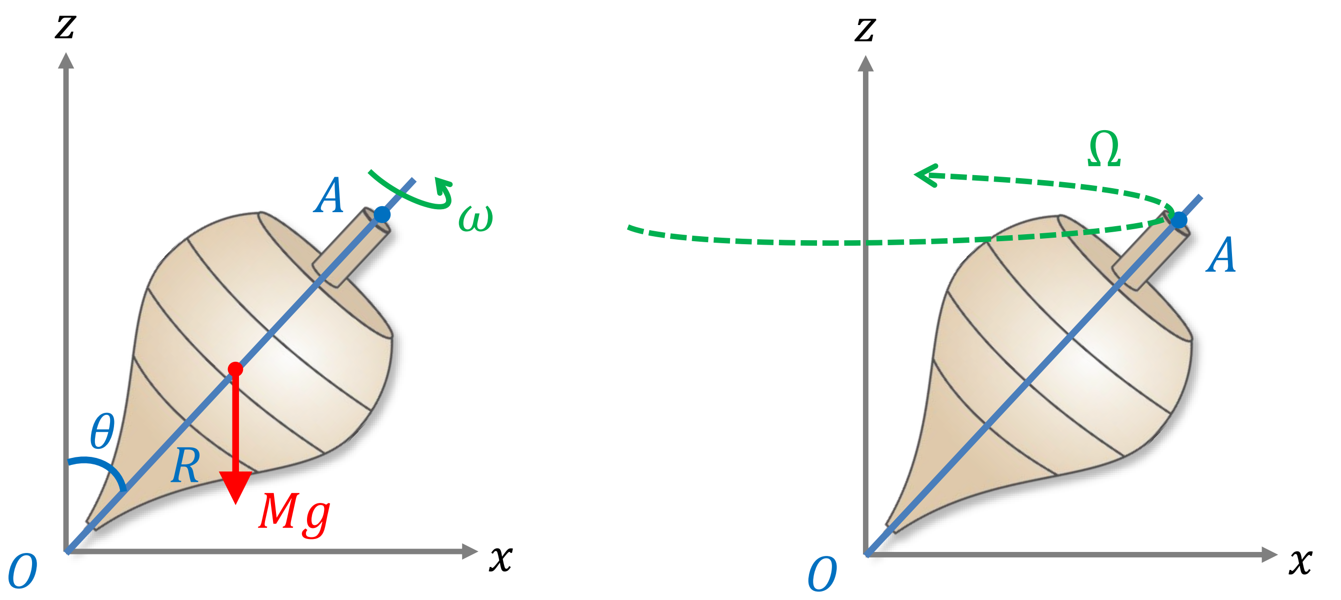


Figure 12a

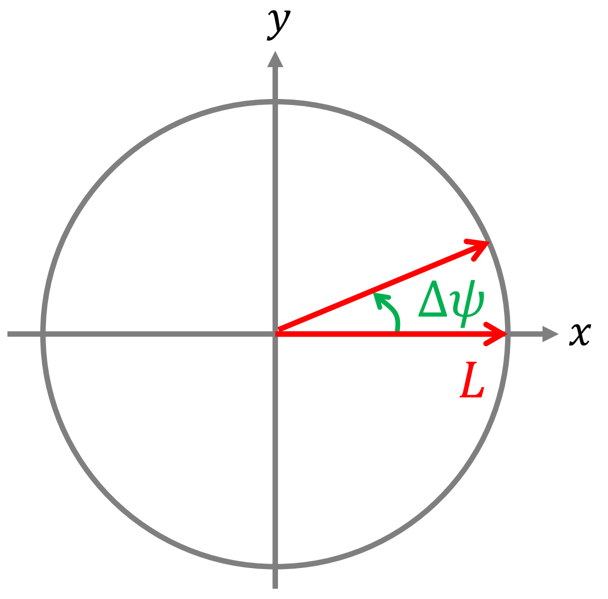
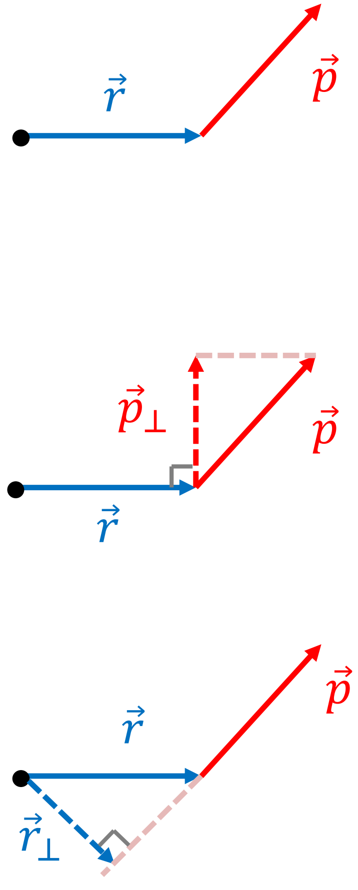


Figure 12b



$$\vec{L} = \vec{r} \times \vec{p}$$

$$L = r\, p_\perp$$

$$L = r_\perp\, p$$

Figure 13

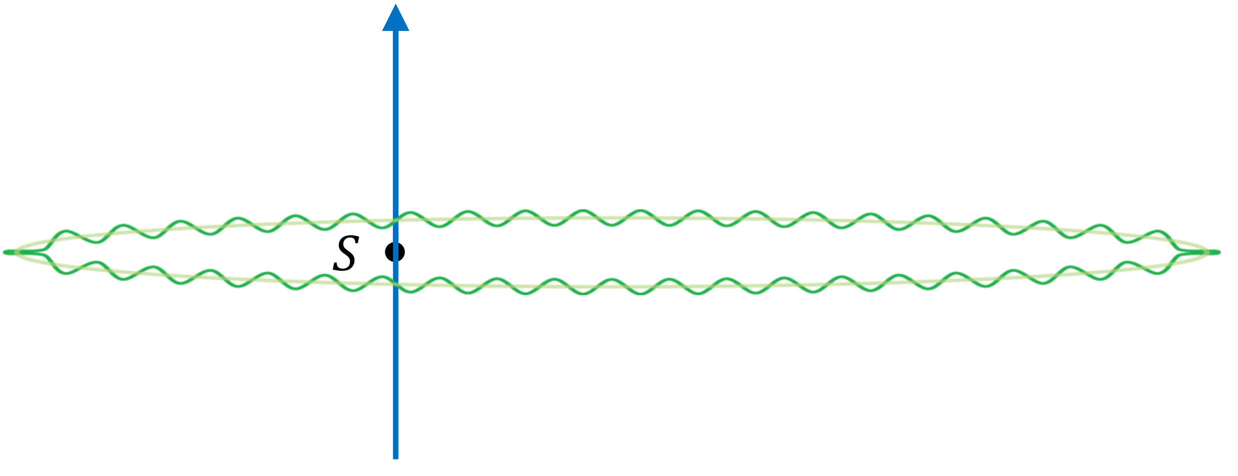


Figure 14

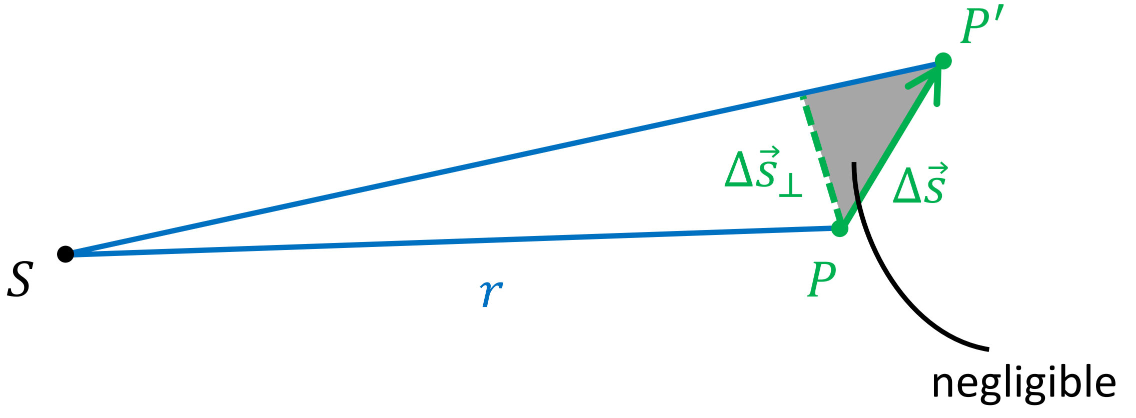


Figure 15

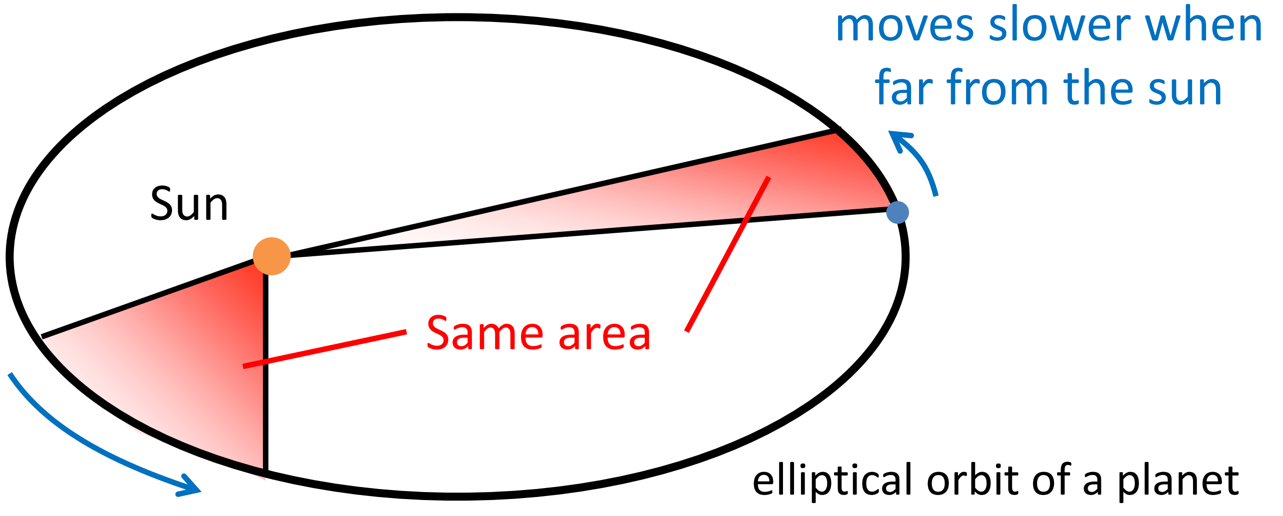

moves slower when
far from the sun
Sun
Same area
elliptical orbit of a planet

Figure 16

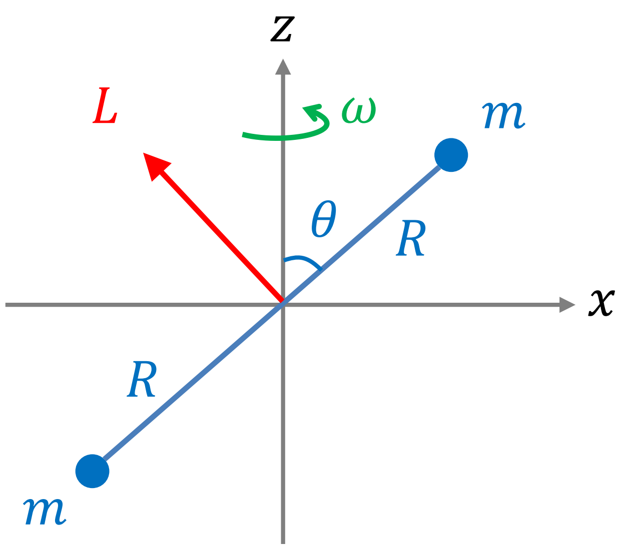


Figure 17a

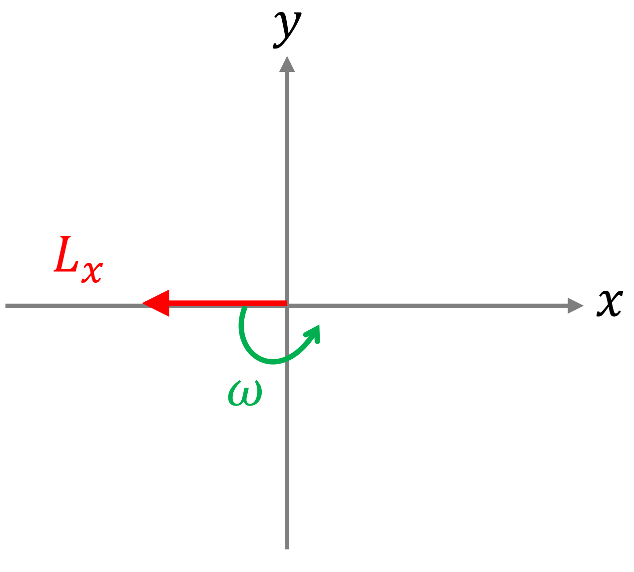


Figure 17b

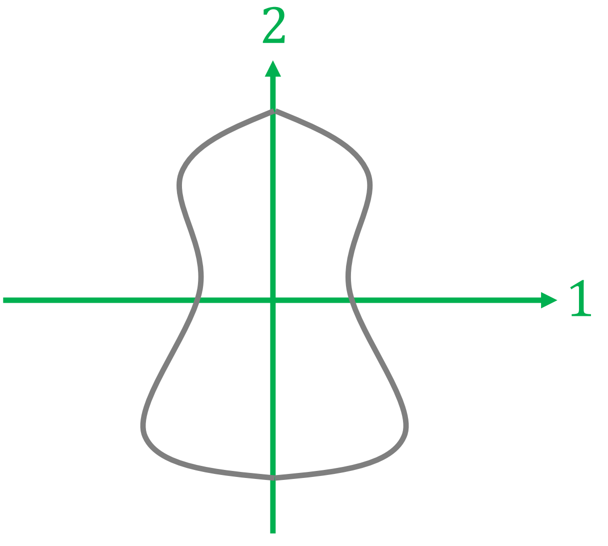


Figure 18a

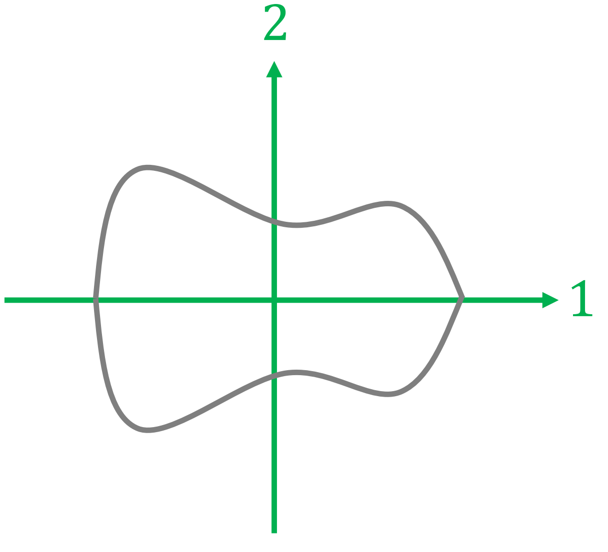


Figure 18b

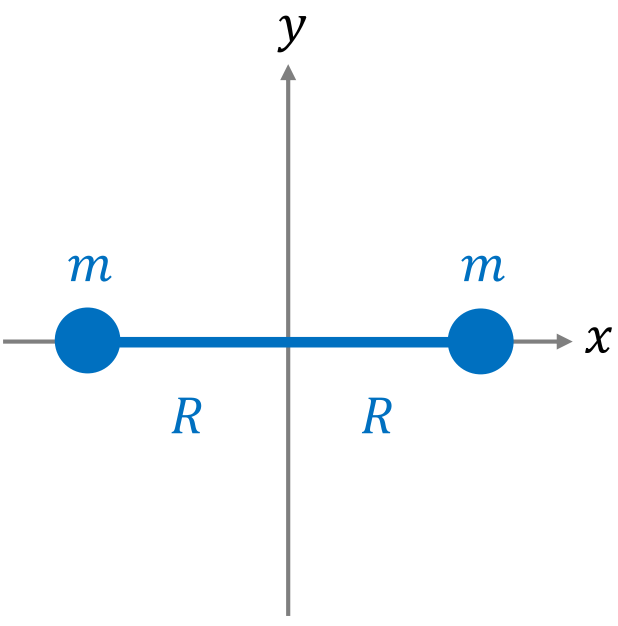


Figure 19a

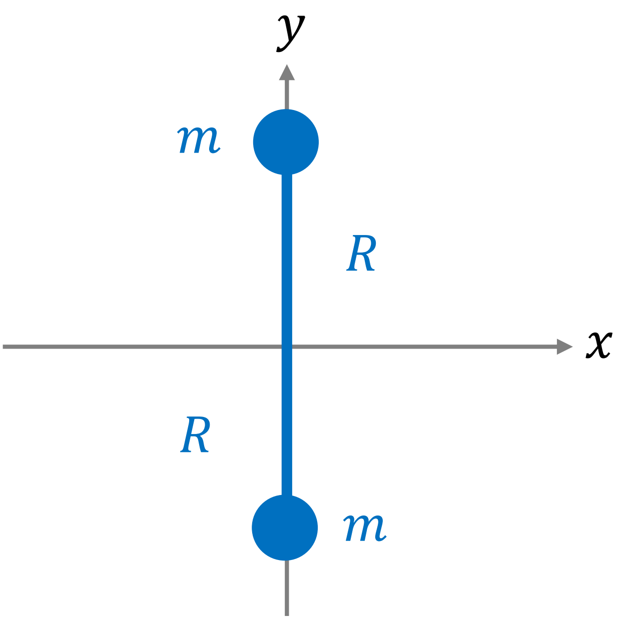


Figure 19b

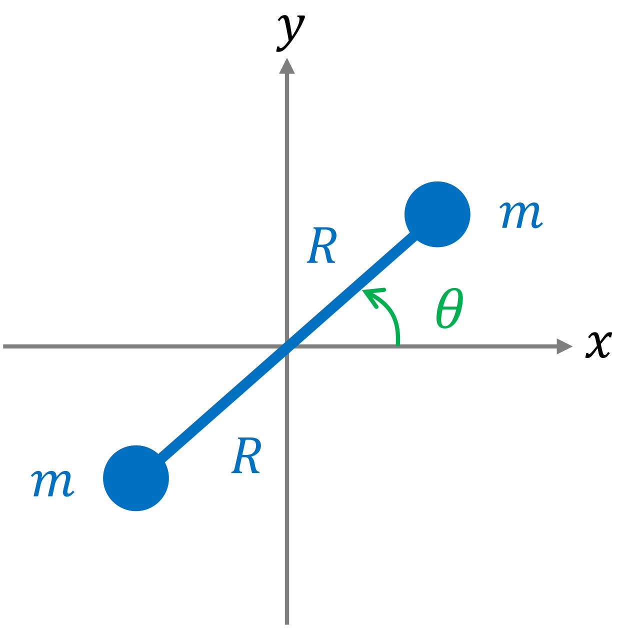


Figure 19c