Moment of inertia

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A systematic account is given of the concept and the properties of the moment of inertia.

Contents

1 Introduction and simple cases

1.1 Introduction

The moment of inertia I is a key concept for rotations. In this elementary discussion, we consider rotation about a fixed axis. If there are masses m_α at positions $\vec{r}_\alpha$, then

$$I = \sum_\alpha m_\alpha r^2_{\alpha,\perp} \tag{1}$$

where $\vec{r}_{\alpha,\perp}$ is the perpendicular distance from the axis to the mass; e.g., for the z-axis,

$$r^2_{\alpha,\perp} = x^2_\alpha + y^2_\alpha \tag{2}$$

It is only necessary to specify the axis but not the origin on the axis; in **Figure 1**, the origins O and O' would give the same moment of inertia, since the perpendicular components of $\vec{r}_\alpha$ are the same; only the parallel component is changed when the origin is shifted along the axis.

It is interesting that the CM involves expressions of the schematic form $\sum mx$ while $I \sim \sum mx^2$.

The moment of inertia I is the analog of the mass m for rotational motion. For example, the angular momentum is

$$L = I\omega \tag{3}$$

where ω is the angular velocity. By analogy with linear motion, I denotes a kind of *inertia* (hence its name) for rotational motion.

But unlike the mass, an object can have different moments of inertia about different axes, and if necessary we shall add a subscript, e.g., I_A, to denote the moment of inertia about a particular axis A. Also the moment of inertia may change if an object is not rigid and changes shape.

If an object has a mass M and typical length scale R (transverse to the axis), then obviously

$$I = \beta M R^2 \tag{4}$$

where β is a pure number of $O(1)$.

The rest of this module is about properties of the moment of inertia, and ways to calculate the value.

1.2 Examples

Example 1

Two point masses, each m, are at the ends of a light rod of length R (**Figure 2**). What is the moment of inertia through an axis perpendicular to the rod and (a) through one end (I_A), and (b) through the CM (I_C)? Express the answers in terms of the total mass $M = 2m$ and R.

$$\begin{aligned} I_A &= (m)(0) + (m)(R)^2 \\ &= mR^2 = \frac{1}{2}MR^2 \\ I_C &= (m)(R/2)^2 + m(R/2)^2 \\ &= \frac{1}{2}mR^2 = \frac{1}{4}MR^2 \end{aligned} \tag{5}$$

Comparing parallel axes, the moment of inertia is smallest for an axis through the CM. We shall discuss this more formally in the next Section. §

Problem 1
An H_2O molecule consists of the O atom (mass m) connected to two H atoms (each of mass μ) by bonds of length R. The angle between the two bonds is 2θ. The total mass is $M = m + 2\mu$. Define $\epsilon = 2\mu/M$.
(a) Find the moment of inertia $I_A = \beta_A MR^2$ about an axis A perpendicular to the plane of the molecule and through the O atom. Answer: $\beta_A = \epsilon$.
(b) Find the distance y between the CM and the O atom.
(c) Find the moment of inertia $I_C = \beta_C MR^2$ about an axis C perpendicular to the plane of the molecule and through the CM. Answer: $\beta_C = \epsilon - \epsilon^2 \cos^2\theta$.
(d) Give numerical values for β_A and β_C in the case of H_2O, for which $2\theta = 104.5^o$. §

Problem 2
Four equal masses are at the corners of a square of side R. Find the moment of inertia, in the form βMR^2 (where M is the total mass of the system), for axes perpendicular to the square and (a) through a corner, (b) through the midpoint of one side, and (c) through the CM. §

2 Two theorems

2.1 Parallel axes theorem

Consider a system of particles, not necessarily a rigid body, with total mass M. Let C be the CM and I_C the moment of inertia about an axis through C, and I_A be the moment of inertia about another axis *parallel* to the first one (**Figure 3**). For convenience choose the point A on the second axis such that

$$\vec{R}_{AC} \equiv \vec{r}_A - \vec{r}_C \tag{6}$$

is perpendicular to the axes. The *parallel axes theorem* states that

$$\boxed{I_A = I_C + MR_{AC}^2} \tag{7}$$

Proof
In the following, without danger of confusion, we simplify the notation: $\vec{R}_{AC} \mapsto \vec{R}$. Choose A to be the origin, and let the position of m_α measured from the origin be $\vec{r}_\alpha$; let the position of m_α measured from C be $\vec{r}_\alpha{}'$. Then

$$\begin{aligned} \vec{r}_{\perp,\alpha} &= \vec{R} + \vec{r}_{\perp,\alpha}{}' \\ \vec{r}_{\perp,\alpha}{}^2 &= \vec{R}^2 + 2\vec{R}\cdot\vec{r}_{\perp,\alpha}{}' + \vec{r}_{\perp,\alpha}{}'^2 \end{aligned} \tag{8}$$

(We have focused on the perpendicular components, though the above statement is also true for the parallel component. Also, $\vec{R}$ is the same as $\vec{R}_\perp$, given our convention on how to define the point A.) Multiply by m_α and sum over α. The LHS is I_A. On the RHS, the first term gives $(\sum_\alpha m_\alpha)R^2 = MR^2$, and the last term gives $\sum_\alpha m_\alpha r_\alpha{}'^2 = I_C$. So the theorem is proved if the cross term is zero:

$$\sum_\alpha m_\alpha \vec{r}_{\perp,\alpha}{}' = 0 \tag{9}$$

which is just the property that the moment about the CM is zero; see the module on the center of mass.

Problem 3
Check that the two results in Example 1 agree with this theorem. §

Problem 4
Check that the results in Problem 1 and Problem 2 agree with this theorem. §

Problem 5
For a thin circular ring of mass M and radius R, find the moment of inertia about the following axes perpendicular to the ring:
(a) an axis through the center;
(b) an axis through a point on the rim. §

2.2 Perpendicular axes theorem

The *perpendicular axes theorem* refers to a lamina, i.e., a sheet with negligible thickness, say in the x–y plane (**Figure 4**). Consider the moments of inertia I_x, I_y, I_z along the three perpendicular axes, all through the same point O on the lamina. The moments of inertia depend on the *perpendicular* distances to the axes; for example, I_z depends on $x^2 + y^2$:

$$I_x = \sum_\alpha m_\alpha \left(y_\alpha^2 + z_\alpha^2\right) = \sum_\alpha m_\alpha y_\alpha^2$$

$$
\begin{aligned}
I_y &= \sum_\alpha m_\alpha \left(x_\alpha^2 + z_\alpha^2\right) = \sum_\alpha m_\alpha x_\alpha^2 \\
I_z &= \sum_\alpha m_\alpha \left(x_\alpha^2 + y_\alpha^2\right) \qquad (10)
\end{aligned}
$$

where in the first two equations we have used the fact that $z_\alpha = 0$ for a thin lamina in the x–y plane. It therefore follows that

$$
\boxed{I_z = I_x + I_y \text{ (lamina in } x\text{–}y \text{ plane)}} \qquad (11)
$$

Example 2
What is the moment of inertia of a thin ring of mass M and radius R (a) about a diameter, and (b) about an axis P parallel to a diameter, but passing through a point on the ring?
(a) Let the ring lie in the x–y plane. For an axis through the center of the ring and perpendicular to the plane, $I_z = MR^2$, since any element of the ring is at the same perpendicular distance R from the axis. By the perpendicular axes theorem, $I_x + I_y = I_z$, and by symmetry $I_x = I_y$. So $I_x = I_y = (1/2)MR^2$.
(b) Applying the parallel axes theorem, we get $I_P = (1/2)MR^2 + MR^2 = (3/2)MR^2$. §

Example 3
Find the moment of inertia of a thin spherical shell of mass M and radius R, about a diameter.

Consider three axes through the center. In obvious notation:

$$
\begin{aligned}
I_x &= \sum_\alpha m_\alpha \left(y_\alpha^2 + z_\alpha^2\right) \\
I_y &= \sum_\alpha m_\alpha \left(z_\alpha^2 + x_\alpha^2\right) \\
I_z &= \sum_\alpha m_\alpha \left(x_\alpha^2 + y_\alpha^2\right) \qquad (12)
\end{aligned}
$$

Adding these together and using the fact that $I_x = I_y = I_z$ by symmetry, we have

$$
\begin{aligned}
3I_x &= 2\sum_\alpha m_\alpha \left(x_\alpha^2 + y_\alpha^2 + z_\alpha^2\right) \\
&= 2MR^2 \\
I_x &= \frac{2}{3}MR^2 \qquad (13)
\end{aligned}
$$

The calculation relies on the following fact: although each of x_α^2, y_α^2 and z_α^2 depends on α, their sum is a constant R^2 independent of α and can be taken out of the sum. §

3 Continuous distribution

For a continuous distribution, the relevant formula takes the form, in obvious notation,

$$
I = \sum \Delta m \, r_\perp^2 \qquad (14)
$$

which is then turned into a suitable integral. We illustrate with several examples.

Example 4: Uniform rod
For a uniform rod of mass M and length R, find the moment of inertia about an axis A through one end, and an axis C through the center (**Figure 5**).

Let the mass per unit length be $\rho = M/R$ and first choose the origin at one end. Then for a segment from x to $x + \Delta x$,

$$
\begin{aligned}
\Delta m &= \rho \, \Delta x \\
I_A &= \sum \Delta m \, x^2 = \sum (\rho \Delta x) \; x^2 \\
&= \rho \int x^2 \, dx = \rho \cdot \frac{1}{3} x^3 \qquad (15)
\end{aligned}
$$

The integral is to be evaluated between $x = 0$ and $x = R$, thus giving

$$
I_A = \rho \frac{1}{3} R^3 = \frac{1}{3} MR^2 \qquad (16)
$$

For the axis C, the calculation is the same, except that the origin is taken to be C, and the integral (16) is to be evaluated between $x = -R/2$ and $x = R/2$, giving

$$
I_C = \; = \frac{1}{12} MR^2 \qquad (17)
$$

It is left as an exercise to check that $I_A - I_C$ agrees with the parallel axes theorem. §

Example 5: Uniform disk
For a uniform circular disk of mass M and radius R, find the moment of inertia about an axis through the center and perpendicular to the disk.

Let the mass per unit area be $\rho = M/(\pi R^2)$. Cut the disk into thin rings, each extending from r to $r + \Delta r$. The mass of this ring is

$$
\Delta m = \rho \cdot 2\pi r \, \Delta r \qquad (18)
$$

and its moment of inertia is

$$
\begin{aligned}
\Delta I &= \Delta m \, r^2 = (\rho \cdot 2\pi r \, \Delta r) \; r^2 \\
&= 2\pi\rho \cdot r^3 \, \Delta r \qquad (19)
\end{aligned}
$$

Adding these up, we find the moment of inertia of the disk to be

$$
\begin{aligned}
I = \sum \Delta I &= 2\pi\rho \sum r^3 \Delta r \\
&= 2\pi\rho \int_0^R r^3\, dr = 2\pi\rho \cdot \frac{1}{4} R^4 \\
&= \frac{1}{2}\left(\rho \cdot \pi R^2\right) R^2 = \frac{1}{2} M R^2 \qquad (20)
\end{aligned}
$$

The same answer applies to a uniform cylinder about its symmetry axis. §

Problem 6
Find the moment of inertia of a uniform disk of mass M and radius R (a) about a diameter, and (b) about an axis perpendicular to the disk and passing through the rim. §

Problem 7
Consider a uniform solid sphere of mass M and radius R. Show that the moment of inertia about a diameter is $(2/5)MR^2$. Hint: cut into thin shells and use Example 3 for each shell. §

Problem 8: Polytropic star of index $n = 1$*
** This is a more complicated example to illustrate how one deals with cases of non-uniform density. It should be skipped when first studying this subject.*

A spherically symmetric star of radius R has a density profile given by

$$\rho(r) = \rho_c\, \theta \qquad (21)$$

where ρ_c is the central density,

$$\theta = \frac{\sin\xi}{\xi} \qquad (22)$$

and $\xi \propto r$ is dimensionless.
(a) The density falls to zero at the surface. Express ξ in terms of r and R.
(b) The moment of inertia about a diameter can be written as $I = \beta M R^2$. Show that

$$\beta = \frac{2}{3\pi^2}\frac{C_3}{C_1} = 0.261 \qquad (23)$$

where

$$C_n \equiv \int_0^\pi \xi^n \sin\xi\, d\xi \qquad (24)$$

Hint: Change integral from dr to $d\xi$. By the way, $\beta < 2/5$ because the mass is more concentrated near the center compared to a uniform sphere. §

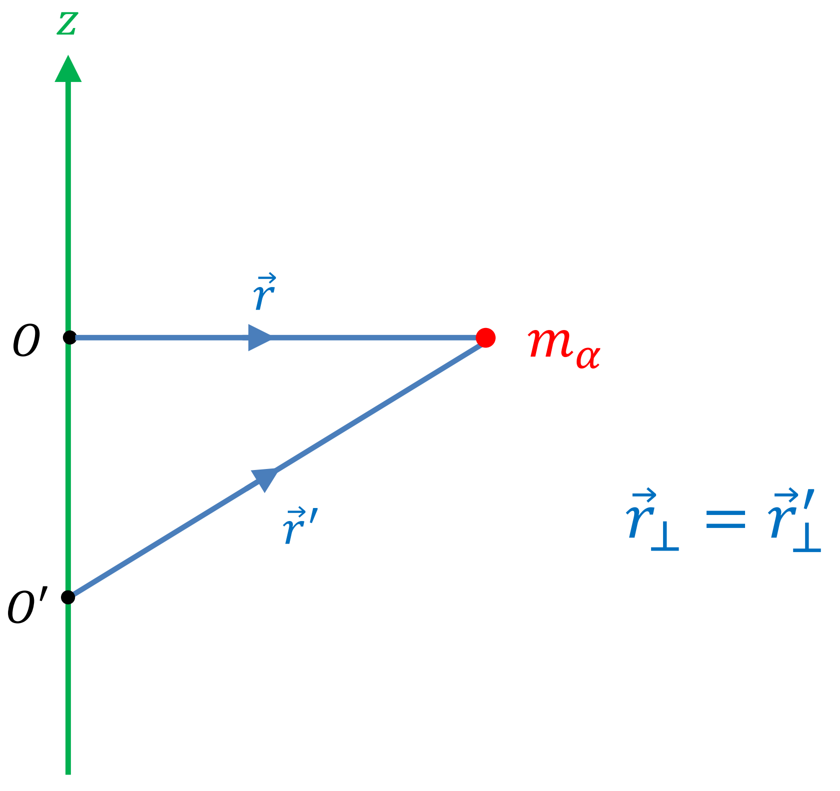


Figure 1

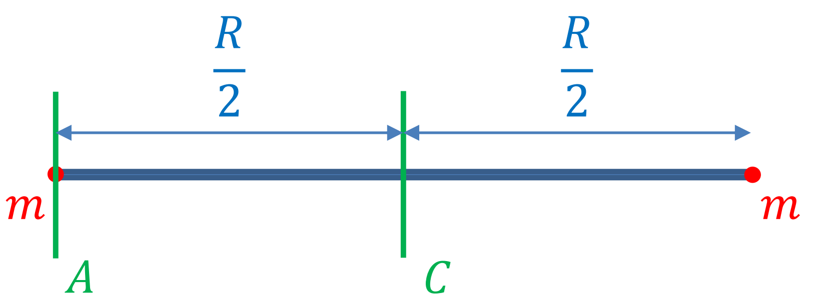
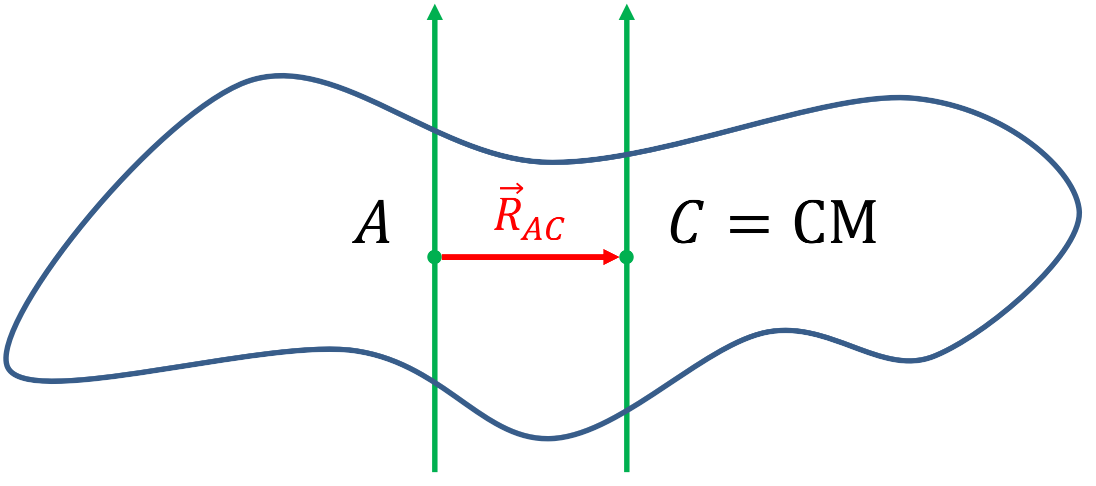


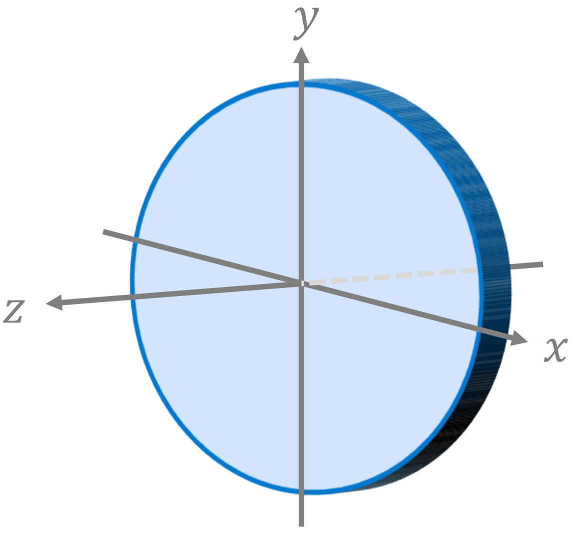
Figure 2



$$I_A = I_C + MR_{AC}^2$$

Figure 3

Lamina in x-y plane



$$I_z = I_x + I_y$$

Figure 4

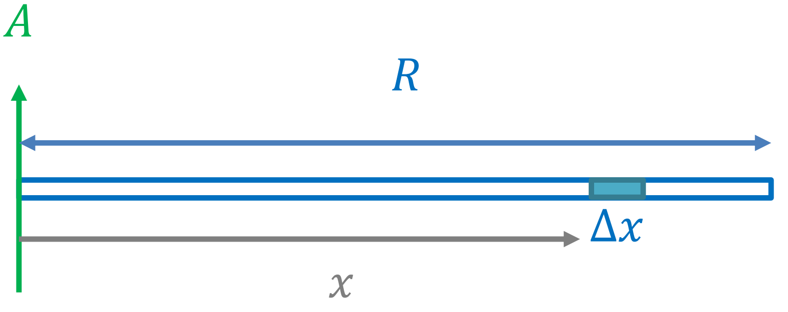


Figure 5