# Center of mass

January 14, 2016

*A systematic account is given of the concept and the properties of the center of mass.*

## Contents



# 1 Introduction

The center of mass (CM) is important in its own right, and as preparation for some aspects of rotations: (a) often we discuss rotations about the CM, and (b) the moment of inertia $I$ is described in a formalism that is in some sense parallel. The position of the CM involves sum of quantities such as $mx$; the moment of inertia (see next module) involves sums of quantities such as $mx^2$.

This Section introduces the following properties of the CM.

- The CM coordinates are the weighted average of the particle coordinates.
- A uniform gravitational field appears to act through the CM — the CM is the same as the center of gravity (CG).
- The moment about the CM is zero.
- The CM of a system of particles (whether or not constituted as a rigid body) behaves as a point particle, with motion controlled by the net external force.

# 2 Position of CM

## 2.1 Weighted average

We begin by reviewing the concept of *average* and *weighted average*.

Suppose there are 3 students with heights (in meters) $x_1 = 1.6$, $x_2 = 1.7$ and $x_3 = 1.8$. Their average height, denoted as $X$, is given by

$$X = \frac{1.6 + 1.7 + 1.8}{1 + 1 + 1} \tag{1}$$

The general formula for the average of $\mathcal{N}$ values $x_1, x_2, \ldots, x_{\mathcal{N}}$ is given by

$$X = \frac{\sum_\alpha x_\alpha}{\sum_\alpha 1} = \frac{\sum_\alpha x_\alpha}{\mathcal{N}} \tag{2}$$

where all sums are understood to be for $\alpha = 1, \ldots, \mathcal{N}$, where $\mathcal{N}$ is the number of individuals.

Next suppose there are 9 students with heights 1.6, 1.6; 1.7, 1.7, 1.7; 1.8, 1.8, 1.8, 1.8. We can of course use (2) with $\mathcal{N} = 9$ terms. It is however simpler to separate them into only 3 groups for $x_\alpha = 1.6, 1.7, 1.8$, but with *weights* of $m_\alpha = 2, 3, 4$ attached to the three cases:

$$X = \frac{2 \times 1.6 + 3 \times 1.7 + 4 \times 1.8}{2 + 3 + 4} \tag{3}$$

In general,

$$X = \frac{\sum_\alpha m_\alpha x_\alpha}{\sum_\alpha m_\alpha} \tag{4}$$

All sums are understood to be over $\alpha = 1, \ldots, N$, where $N$ is the number of groups. An average such as (4) is called a *weighted average.*

**Problem 1**
Take the example above, and calculate $X$
(a) by summing over $\mathcal{N} = 9$ individual cases as in (2), and
(b) by summing over $N = 3$ groups with weights as in (4). §

## 2.2 Definition of CM

**Figure 1** shows three masses with $m_\alpha = 2, 3, 4$ (say in kg) at positions $x_\alpha = 1.6, 1.7, 1.8$ (say in m). The coordinate $X$ of the center of mass (CM) is defined in exactly the same way as in (3). The general definition is the same as (4). Generalizing to 3D, the position of the CM is

$$\vec{R} = X\,\hat{\mathbf{i}} + Y\,\hat{\mathbf{j}} + Z\,\hat{\mathbf{k}} \tag{5}$$

with each component given by a formula such as (4), or combining them together,

$$\vec{R} = \frac{\sum_a m_\alpha \vec{r}_\alpha}{\sum_\alpha m_\alpha} = \frac{1}{M} \sum_\alpha m_\alpha \vec{r}_\alpha \tag{6}$$

where

$$\vec{r}_\alpha = x_\alpha\,\hat{\mathbf{i}} + y_\alpha\,\hat{\mathbf{j}} + z_\alpha\,\hat{\mathbf{k}} \tag{7}$$

is the vector position of mass $\alpha$, and

$$M = \sum_\alpha m_\alpha \tag{8}$$

is the total mass.

**Problem 2**
Three masses are at the corners of a triangle in the $x$–$y$ plane, as shown in **Figure 2** and as summarized in Table 1.

| $\alpha$ | $m_\alpha$ | $x_\alpha$ | $y_\alpha$ |
|---|---|---|---|
| 1 | 3 | 0 | 0 |
| 2 | 4 | 1 | 0 |
| 3 | 5 | 1/2 | $\sqrt{3}/2$ |

Table 1. Three point masses at three positions.

Find the position of the CM. §

**Problem 3**
An $H_2O$ molecule consists of an O atom ($m = 16$) connected to two H atoms ($m = 1$) with bonds of length $R = 0.096$ nm. The two bonds make an angle of $2\theta = 104.5^\circ$ with each other. Find the distance between the CM and the O atom. (Regard the atoms, or strictly speaking the nuclei containing most of the masses, as points.) §

**Problem 4**
A double decker bus including its passengers has a mass of 15 000 kg. One passenger, of mass 60 kg, goes from the lower deck to the upper deck, which are 2.5 m different in height. By how much has the CM of the system shifted upwards? §

**Problem 5**
Consider the sun and the earth. How far is the CM of the system from the center of the sun? Look up the necessary data. §

There is no assumption that the various masses $m_i$ are connected (e.g., Problem 4) or that they have no relative motion (e.g., Problem 5). However, if the masses are parts of a rigid body, then the CM is a fixed point on that body.

# 3 Continuous distribution

For many continuous distributions of mass, the position of the CM can often be determined (fully or partially) by symmetry. So for the examples below, we do not have to do a full calculation for all three coordinates of the CM.

**Example 1**
Find the CM of an arbitrary triangular lamina of uniform density.

Consider the triangle $ABC$ in **Figure 3a**. Regard it as the sum of thin strips as shown in **Figure 3b**. The CM of each strip (a thin rectangle) is at its midpoint, so the whole lamina can be regarded as a series of point masses on the line $AP$, where $P$ is the midpoint of $BC$; thus $AP$ is a *median.* By the same argument the CM must also lie on the other median $BQ$ (**Figure 3c**). Thus the CM is at the *centroid*, where the medians meet. By the way, this gives an indirect proof that the three medians meet at the same point (**Figure 3d**). §

**Example 2**
Find the CM of a uniform thin lamina in the shape of a semicircle of radius $R$ (**Figure 4a**).

By symmetry, the CM must lie on the line shown, at a vertical coordinate $Y = \gamma R$, where $\gamma$ is a pure number, with $0 < \gamma < 1$. (In fact, since the lower parts of the lamina are bigger, it is obvious that $0 < \gamma < 1/2$.) Cut the lamina into strips as shown (**Figure 4b**), and let the mass per unit area be $\rho$. The mass of each strip is

$$\begin{aligned} \Delta m &= \rho \times \text{area} \\ &= \rho\,(2x\Delta y) = 2\rho\sqrt{R^2 - y^2}\,\Delta y \end{aligned} \tag{9}$$

where $x$ is the coordinate of the right boundary. Then, using the standard formula

$$\begin{aligned} Y &= \frac{\sum \Delta m \cdot y}{\sum \Delta m} \\ &= \frac{\sum 2\rho\sqrt{R^2 - y^2}\,y\,\Delta y}{\sum 2\rho\sqrt{R^2 - y^2}\,\Delta y} \\ &= \frac{\int \sqrt{R^2 - y^2}\,y\,dy}{\int \sqrt{R^2 - y^2}\,dy} \end{aligned} \tag{10}$$

In the last step, we have cancelled a common factor of $2\rho$ and converted the sum to an integral, understood to be between the limits $y = 0$ and $y = R$. In these expressions, the numerator always has an extra factor of $y$.

It is convenient to change to the dimensionless variable $\xi$ by $y = \xi R$. The rest of the calculation is left as a problem.

**Problem 6**
Continue with this Example.
(a) Show that

$$\gamma = \frac{\int \sqrt{1 - \xi^2}\,\xi\,d\xi}{\int \sqrt{1 - \xi^2}\,d\xi} \tag{11}$$

where the integrals are over $0 < \xi < 1$.
(b) Evaluate these integrals and find $\gamma$.
(c) Check that $0 < \gamma < 1/2$. §

**Problem 7**
The CM of a uniform solid hemisphere of radius $R$ is a height $Y = \gamma' R$ from the base.
(a) Without doing any calculations, would you say $\gamma'$ is larger or smaller than $\gamma$ in the last problem?
(b) Actually evaluate $\gamma'$. §

**Problem 8**
Go back to the triangular lamina of Example 1 and suppose $A$ is at a height $h$ above the base $BC$. If the height of the CM is $Y = \gamma h$, find the dimensionless number $\gamma$. §

# 4 Equilibrium and the center of gravity

## 4.1 Equivalent force

A number of masses $m_\alpha$ are attached to a horizontal massless rod, at coordinates $x_\alpha$, and placed in a *uniform* gravitational field, so vertical forces $m_\alpha g$ act on the masses (**Figure 5a**). It is claimed that the situation is equivalent to a single force $Mg$ acting through the CM (**Figure 5b**), where $M = \sum_\alpha m_\alpha$. Equivalence means that the net force and the net torque are the same. The former is obvious, and we now verify the latter, about an axis through the origin $O$ and perpendicular to the page.

The force $m_\alpha g$ with moment arm $x_\alpha$ produces a torque $\tau_\alpha = (m_\alpha g)x_\alpha$; the total torque in **Figure 5a** is

$$\tau = \sum_\alpha \tau_\alpha = \sum_\alpha m_\alpha g x_\alpha \tag{12}$$

For the situation in **Figure 5b**, the total torque is

$$\tau' = (Mg)X \tag{13}$$

These two expressions are equal, by the definition of $X$ in (4).

The point through which gravity appears to act is called the *center of gravity* (CG). In a uniform gravitational field, the CG is the same as the CM. The proof is readily generalized to masses not confined to one axis.

Note that these results depend on two assumptions: (a) the gravitational mass is the same as the inertial mass; and (b) The gravitational field strength is uniform.

**Problem 9**
Check that the two ways of calculating the torque still agree if the axis is taken to be at $x = c$ rather than the origin. §

## 4.2 Conditions for equilibrium under uniform gravity

It was shown in the last module that the condition for rotational equilibrium is that the net torque is zero. Here we apply it to equilibrium under uniform gravity, i.e., where the acceleration due to gravity has the same direction and the same magnitude $g$.

The condition can be stated very simply: A body placed in a uniform gravitational field will be in equilibrium if supported (e.g., hung or pivoted) at the CM.

The proof is simple. The forces on the different parts of the system can be replaced by a single force $Mg$ acting at the CG (= CM). Thus, if the supporting force (e.g., due to a string or a pivot) acts through the same point (**Figure 6**), the two forces together will produce zero torque about the CM, and hence also about the any other point.

**Example 3**
Two masses $m_1$, $m_2$ are placed at the ends of a light rod $AB$ of length $\ell$. The rod is supported at the point $P$ where $AP = r_1$, $PB = r_2$ (**Figure 7a**). How should $r_1$ and $r_2$ be related if the system is in equilibrium?

The solution is obvious, namely

$$m_1 r_1 = m_2 r_2 \tag{14}$$

This trivial example is included to set the stage for the next Section. §

**Example 4**
A rectangular block (not necessarily uniform vertically) has a base $AB = 2b$, and its CG is a height $h$ (not necessarily half the total height) from the base (**Figure 8a**). It is placed on a slope with an angle $\theta$ (**Figure 8b**). Assume there is enough friction so that the block does not slide down. What is the maximum value of $\theta$ such that the block does not topple?

On the point of toppling, the block will balance on one corner $A$. To remove reference to the unknown force $F$ exerted at $A$ (which is a combination of normal reaction and friction), we consider the torque about this point. The gravitational force must act along the a line passing through $A$ — if it passes "outside" the base, there will be a net torque in the direction to cause the block to topple. Therefore we get

$$\tan\theta = \frac{b}{h} \tag{15}$$

for the limiting angle $\theta$. If the CM is low, a large angle is possible. §

**Example 5**
A double-decker bus has a base (distance between the left and right wheels) of $2b = 2.4$ m. Its CG is a height $h = 1.2$ m from the ground. Its total mass is $M = 15\,000$ kg. It is travelling at a speed of 36 kph and making a left turn on a road with radius of curvature $r = 40$ m. What are the normal reaction forces $N_1$ and $N_2$ on the left and the right wheels respectively, assuming there is enough friction so that the bus does not slide sideways off the road? Refer to **Figure 9a**. The front and rear wheels on each side are considered together.

We first list the forces (**Figure 9b**).

- Gravity $W = Mg$ acts through the CM.
- Because the bus is undergoing circular motion, in the frame co-moving with the bus, there is a centrifugal pseudo-force $F = Mv^2/r$ which is horizontal and acting through the CM.
- There are reaction forces $N_1$ and $N_2$ at the two sets of wheels.
- There is friction from the road, just enough to cancel $F$. This is shown as $G$ in the figure. The forces due to the two sets of wheels are combined into one force, because they act along the same line.

There are two conditions. The vertical forces have to balance, so

$$N_1 + N_2 = Mg \tag{16}$$

The total torque about the CM also has to be zero, so

$$-N_1 b - Gh + N_2 b = 0 \tag{17}$$

where counterclockwise torque is taken as positive. Simplifying,

$$N_2 - N_1 = (h/b)G = (h/b)Mv^2/r \tag{18}$$

Hence we have

$$N_{2,1} = \frac{1}{2}Mg\left(1 \pm \frac{h}{b}\cdot\frac{v^2}{gr}\right) \tag{19}$$

where the upper (lower) sign refers to $N_2$ ($N_1$). If the bus is not moving ($v = 0$) or is travelling along a straight road ($r \to \infty$), or if the CG is very low ($h/b \ll 1$), then each set of wheels supports half the weight; otherwise the outside wheels take up more weight. §

**Problem 10**
Put in numbers and evaluate $N_1$, $N_2$ §

**Problem 11**
In general, what is the maximum speed for such a bus when it is going round a bend with a radius of curvature $r$? §

# 5 Moments about CM

## 5.1 An example

Go back to Example 3 and **Figure 7a**. If we take the point $P$ to be the CM, then condition (14) holds. Choose a new set of coordinates $x'$, measured from the CM (**Figure 7b**). Then taking care of signs, $x'_1 = -r_1$, $x'_2 = r_2$, and the condition for equilibrium becomes

$$m_1 x'_1 + m_2 x'_2 = 0 \tag{20}$$

For many masses (**Figure 7c**), the condition is obviously

$$\sum_\alpha m_\alpha x'_\alpha = 0 \tag{21}$$

This means (multiplying throughout by $g$) that there is at much clockwise torque (negative terms) as anticlockwise torques (positive terms). We say that the *net moment about the CM is zero.*

## 5.2 General statement and proof

We now give the general statement in 3D and also provide a formal proof. Consider masses $m_\alpha$ at positions $\vec{r}_\alpha$ measured from an arbitrary origin $O$. The particles need not make up a rigid body; the positions may change with time, and we are only looking at the situation at one time.

The CM is the point $P$, at position $\vec{R}$ given by (6). The position of particle $\alpha$ with respect to $P$ is $\vec{r}_\alpha{}'$, with the relationship (**Figure 10**)

$$\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}' \tag{22}$$

Now multiply by $m_\alpha$ and sum:

$$\sum_\alpha m_\alpha \vec{r}_\alpha = \left(\sum_\alpha m_\alpha\right)\vec{R} + \sum_\alpha m_\alpha \vec{r}_\alpha{}' \tag{23}$$

The LHS is $M\vec{R}$ by (6). For the first term on the RHS, $\vec{R}$ does not depend on $\alpha$ and has been taken out of the summation, and the sum in brackets gives $M$; thus this term is the same as the LHS. Hence the last term is zero, namely

$$\boxed{\sum_\alpha m_\alpha \vec{r}_\alpha{}' = 0} \tag{24}$$

which is the result we want, generalizing (21).

## 5.3 Breaking up the KE

Consider a cylinder rolling down a slope (**Figure 11a**), or a bottle after it has been thrown (**Figure 11b**). We can describe such motions of a rigid body by specifying at any one time (a) the position of the CM, namely $\vec{R}$, and (b) the position of each small particle with respect to the CM, namely $\vec{r}_\alpha{}'$. The change of the former is the motion of the CM; the change of the latter is a rotational motion about the CM.

We claim that the total KE is the sum of two terms: (a) the KE of the CM, regarded as a point mass $M$ moving according to $\vec{R}(t)$, and (b) the rotational KE about the CM. To prove this, differentiate (22) with respect to time, and in obvious notation

$$\vec{v}_\alpha = \vec{V} + \vec{v}_\alpha{}' \tag{25}$$

Square to get

$$v_\alpha^2 = V^2 + 2\vec{V}\cdot\vec{v}_\alpha{}' + v_\alpha{}'^2 \tag{26}$$

Multiply by $(1/2)m_\alpha$ and sum. The LHS is the total KE. On the RHS, the first term gives $(1/2)MV^2$, i.e., the KE of the CM. The last term gives the KE of the motion about the CM; it can be written as $(1/2)I\omega^2$ using the appropriate moment of inertia. So our claim is equivalent to saying that the cross term vanishes, namely

$$\vec{V}\cdot\sum_\alpha m_\alpha \vec{v}_\alpha{}' = 0 \tag{27}$$

which is obvious by differentiating (24).

Warning: This decomposition of the total KE does not work for other axes.

**Problem 12**
A cylinder of mass $M$ and radius $R$ is rolling down an incline, with linear velocity $V$ for the center, and angular frequency $\omega$ about the center (**Figure 12**). The moment of inertia is $I = \beta M R^2$. (For example, $\beta = 1/2$ for a solid cylinder, but we leave $\beta$ general below).
(a) What is the total KE in terms of $M, V, I, \omega$?
(b) Let the point of contact on the cylinder be $Q$. What is the velocity of $Q$ with respect to the center? Hence what is the velocity of $Q$ respect to the slope?
(c) If $Q$ is instantaneously at rest with respect to the surface with which it is in contact, we say there is *no slip*. In this situation, relate $V$ and $\omega$.
(d) Hence express the total KE in terms of $M, V$ and $\beta$ when there is no slip. §

# 6 Motion of the CM

Consider a collection of point masses $m_\alpha$, which need not be parts of a rigid body. Newton's second law for each point mass gives

$$m_\alpha \frac{d^2\vec{r}_\alpha}{dt^2} = \vec{F}_\alpha = \vec{F}^e_\alpha + \sum_\beta \vec{F}_{\alpha\beta} \qquad (28)$$

The total force $\vec{F}_\alpha$ acting on $\alpha$ is the sum of an external force[1] $\vec{F}^e_\alpha$ and internal forces $\vec{F}_{\alpha\beta}$ acting on particle $\alpha$ due to particle $\beta$. The convention is $F_{\alpha\alpha} = 0$ (a particle does not act on itself). Sum the above equation over $\alpha$.

$$\text{LHS} = \sum_\alpha m_\alpha \frac{d^2\vec{r}_\alpha}{dt^2} = \frac{d^2}{dt^2}\left(\sum_\alpha m_\alpha \vec{r}_\alpha\right)$$
$$= \frac{d^2}{dt^2}(M\vec{R}) = M\frac{d^2\vec{R}}{dt^2} \qquad (29)$$

i.e., mass × acceleration for a hypothetical point particle of mass $M$ at $\vec{R}(t)$.

On the right hand side, the internal forces add up to

$$\sum_\alpha \sum_\beta \vec{F}_{\alpha\beta} = 0 \qquad (30)$$

[1] That is, external to the system.

by Newton's third law, e.g., $\vec{F}_{12} + \vec{F}_{21} = 0$. That leaves the net external force, i.e., the sum of the external forces acting on each particle:

$$\vec{F}^e = \sum_\alpha \vec{F}^e_\alpha \qquad (31)$$

Therefore we recover Newton's second law for the motion of the CM:

$$\vec{F}^e = M\frac{d^2\vec{R}}{dt^2} \qquad (32)$$

If an object is thrown, each part of it may have a complicated motion; but if we put a dot on the CM, that dot undergoes very simple motion, e.g., along a parabola if the object is in a uniform gravitational field (**Figure 13**).

**Example 6**
Two masses each of 1 kg lie on a smooth table; they are not connected. A force of 1 N acts on one of them (**Figure 14**). Find the acceleration of each mass and the acceleration of the CM.

For the mass on which the force acts, the acceleration is 1 N / 1 kg = 1 m $s^{-2}$. The other mass remains at rest.

The CM (the mid-point between the two masses) obeys Newton's second law with net force $F = 1$ N and total mass $M = 2$ kg; so it has an acceleration 1 N / 2 kg = 0.5 m $s^{-2}$. The purpose of this example is to stress that the CM can be just a mathematical point — there is nothing there. §

**Problem 13**
A uniform circular disk is thrown with the following initial conditions: horizontal velocity of the CM $u_0 = 3.0$; vertical velocity of the CM $v_0 = 4.0$ m $s^{-1}$; angular velocity abut the CM $\omega_0 = 1.5$ $s^{-1}$(**Figure 15**). The center of the disk is $P$ and the point initially at the top of the disk is $Q$. Find the following after a time of 3.0 s:
(a) the coordinates of $P$ relative to its initial position $P_0$;
(b) the coordinates of $Q$ relative to $P$; and
(c) the coordinates of $Q$ relative to $P_0$. §

The spreadsheet `tumble2.xlsx` shows the positions of $P$ and $Q$ as functions of time, and also plots the trajectories. You should redo the spreadsheet (by just changing one parameter) and see what happens for larger $\omega_0$, e.g., 10 times larger. For small $\omega_0$, there is little rotation and the motions of $Q$

looks very much like $P$, but for larger $\omega_0$, the motion of $Q$ looks complicated, but is in fact a combination of two simple motions.

# Appendix

## A Averages and moments

Many formulas and derivations can be simplified through an abstract notation. Any quantity depending on the index $\alpha$ will be denoted by a lower case symbol, e.g., $f_\alpha$, and then the index is dropped:

$$f_\alpha \quad \mapsto \quad f \tag{33}$$

The average is defined as

$$\langle f \rangle \quad = \quad \frac{\sum_\alpha m_\alpha f_\alpha}{\sum_\alpha m_\alpha} \tag{34}$$

for a given set of weights $m_\alpha$. Note that taking average is a linear operation, and $\langle 1 \rangle = 1$.

For a variable $x$, let

$$X \quad = \quad \langle x \rangle \tag{35}$$

and write

$$x \quad = \quad X + x' \tag{36}$$

Taking average, we see

$$\langle x \rangle \quad = \quad X + \langle x' \rangle \tag{37}$$

proving the last term is zero — namely the moments about the CM is zero.

The same formulas are useful in statistics (for which uniform weights are the most common). It is suggestive to write $x'$ as $\Delta x$, the *deviation* from the mean:

$$\begin{aligned}
\text{deviation} \quad &= \quad \Delta x \\
&= \quad x - X \\
\text{square deviation} \quad &= \quad (\Delta x)^2 \\
&= \quad x^2 - 2xX + X^2 \\
\text{mean square deviation} \quad &\equiv \quad \sigma^2 = \langle (\Delta x)^2 \rangle \\
&= \quad \langle x^2 \rangle - 2\langle x \rangle X + X^2 \\
&= \quad \langle x^2 \rangle - X^2 \\
&= \quad \langle x^2 \rangle - \langle x \rangle^2
\end{aligned} \tag{38}$$

The parameter $\sigma$ is called the *root mean square deviation* or the *standard deviation*.

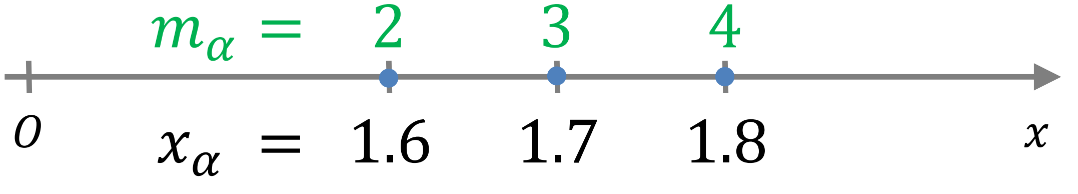



Figure 1

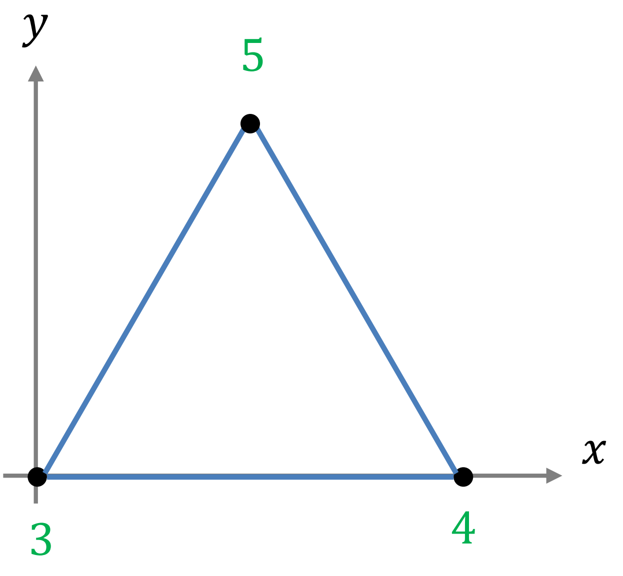



Figure 2

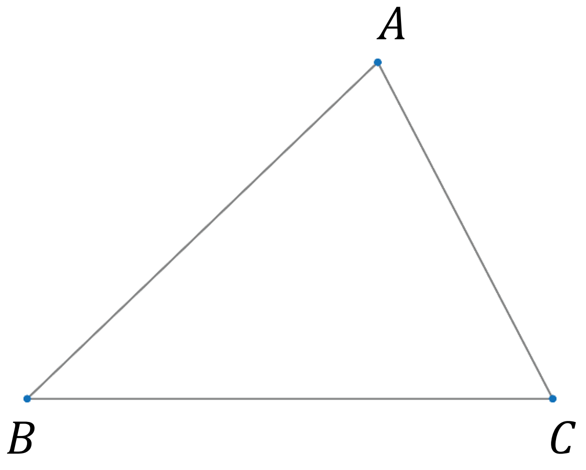



Figure 3a

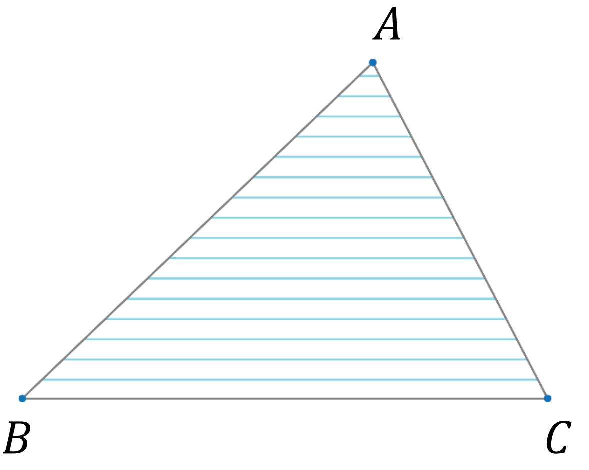



Figure 3b

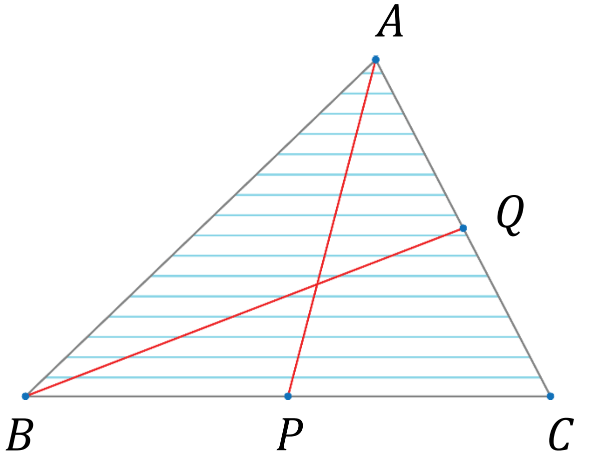



Figure 3c

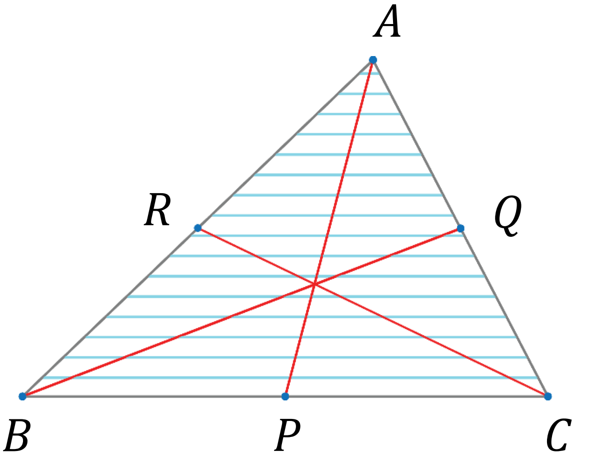



Figure 3d

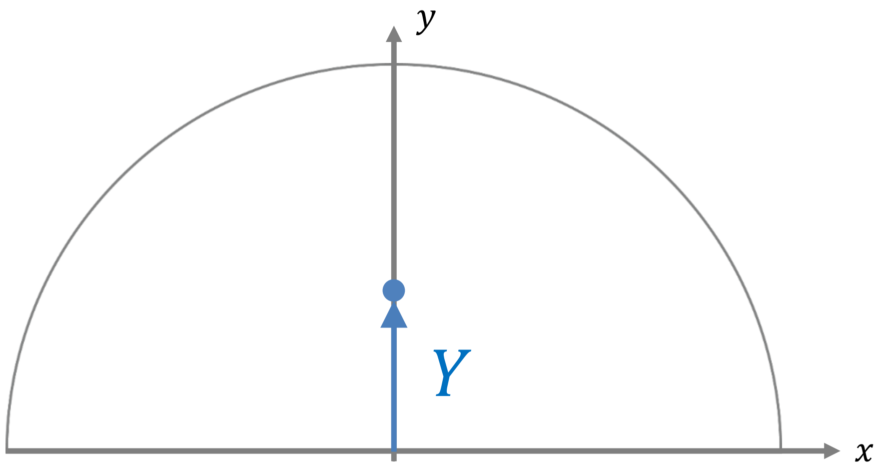



Figure 4a

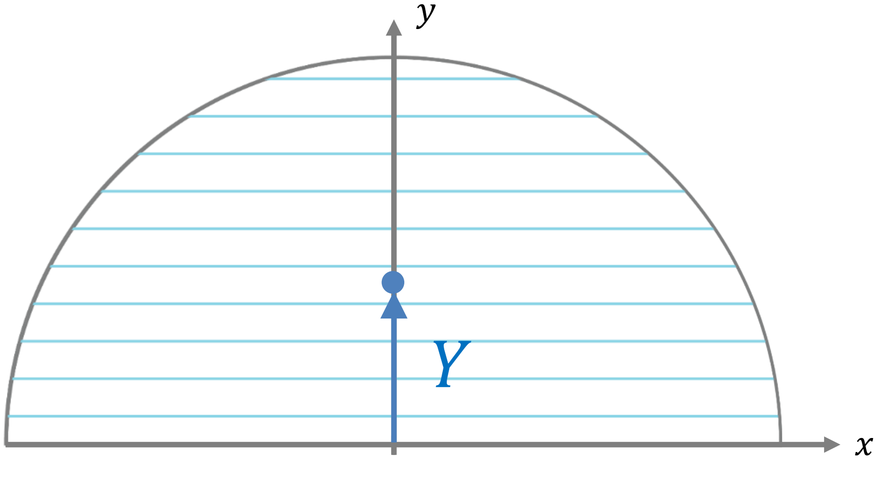



Figure 4b

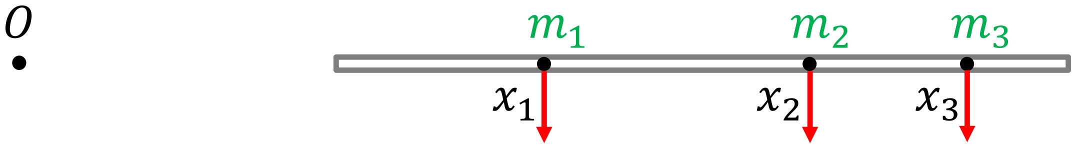



Figure 5a


Figure 5b

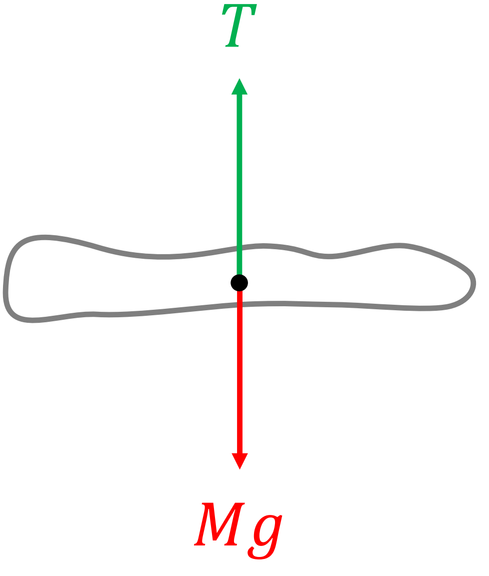



Figure 6

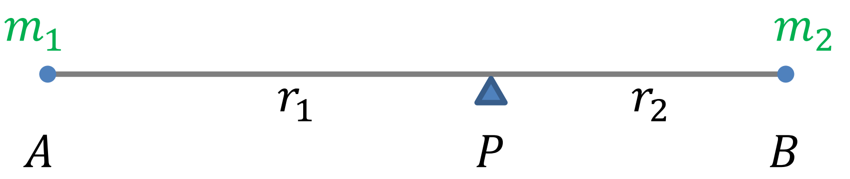



Figure 7a

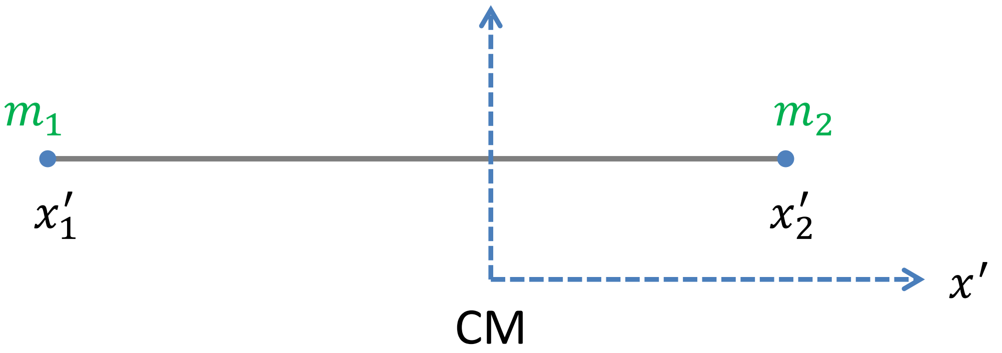



Figure 7b

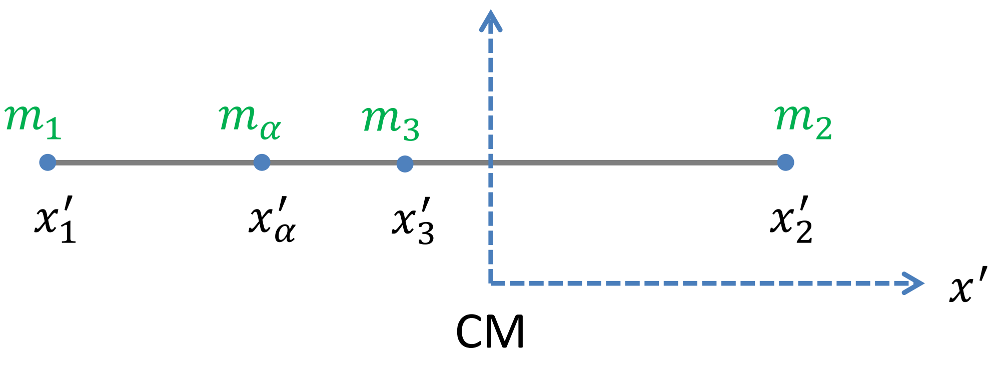



Figure 7c

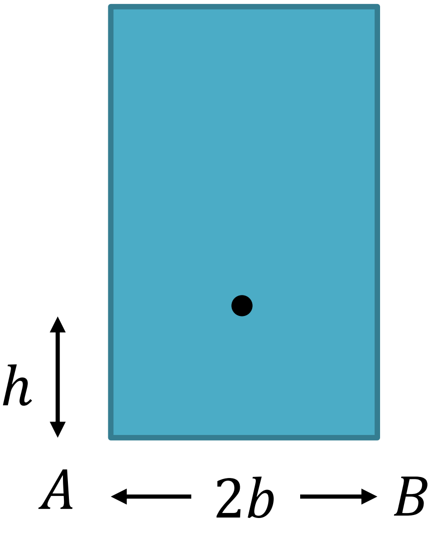



Figure 8a

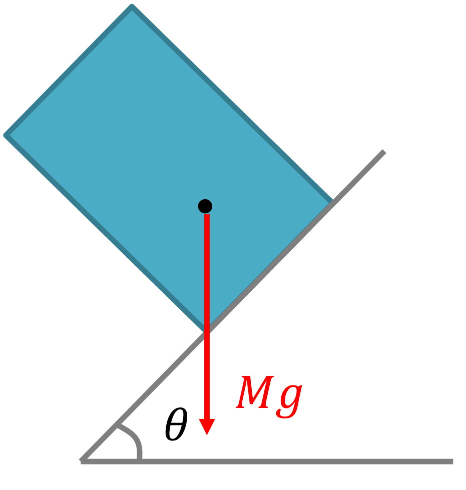
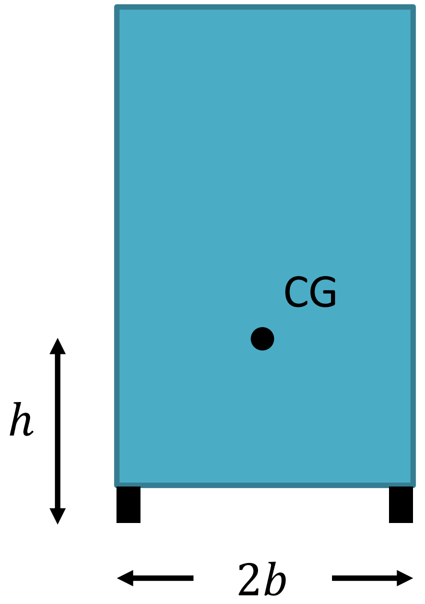



Figure 8b




$2b = 2.4$ m

$h = 1.2$ m

$v = 36$ kph

$r = 40$ m

$M = 15000$ kg

Figure 9a

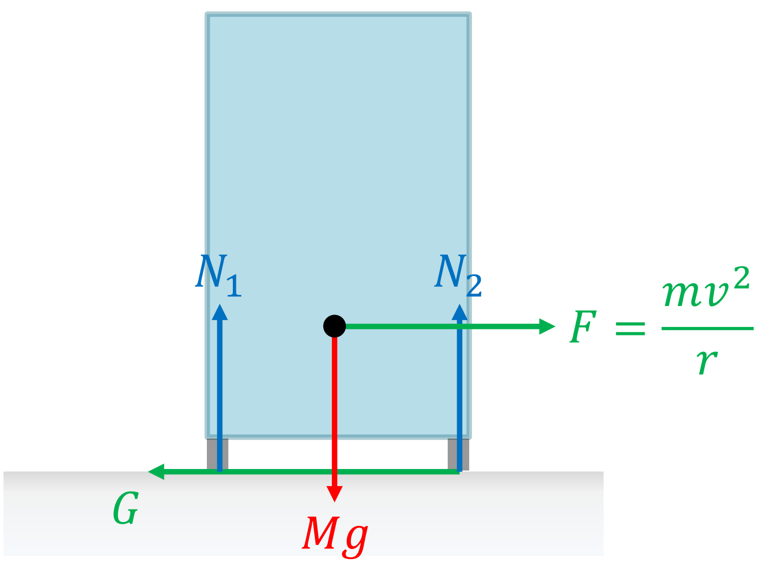
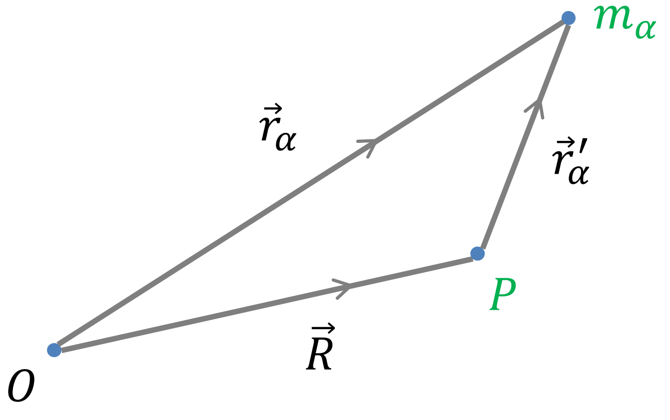



Figure 9b




$$\vec{r}_\alpha = \vec{R} + \vec{r}'_\alpha$$

Figure 10

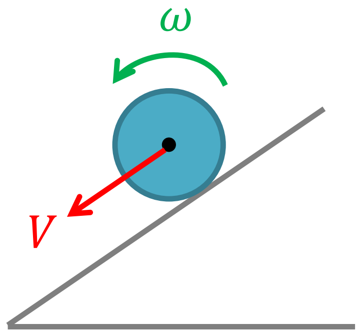



Figure 11a

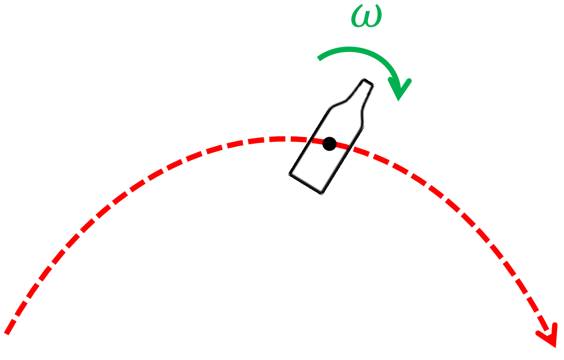



Figure 11b

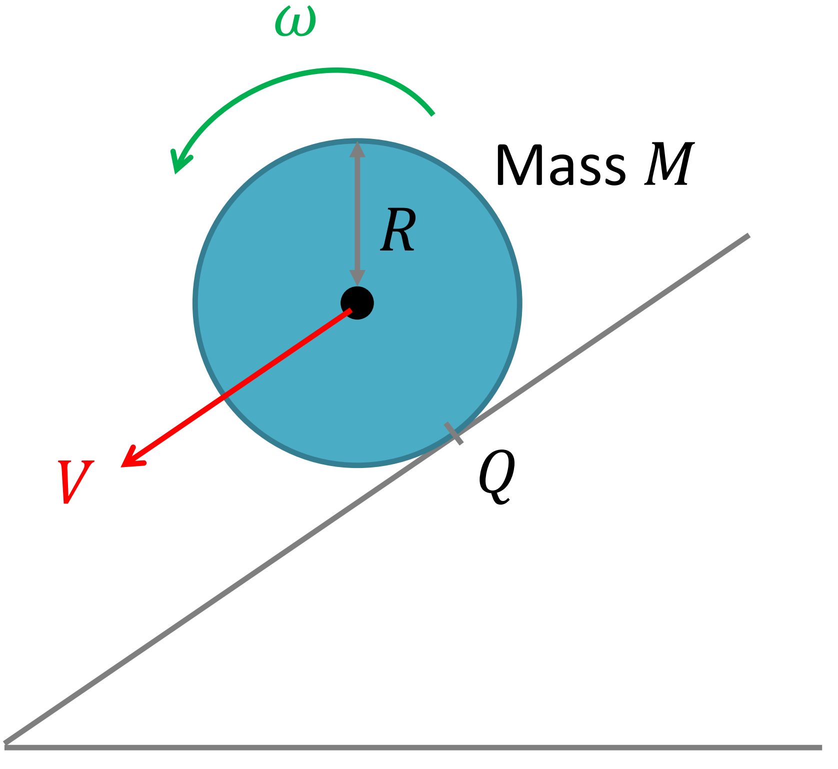
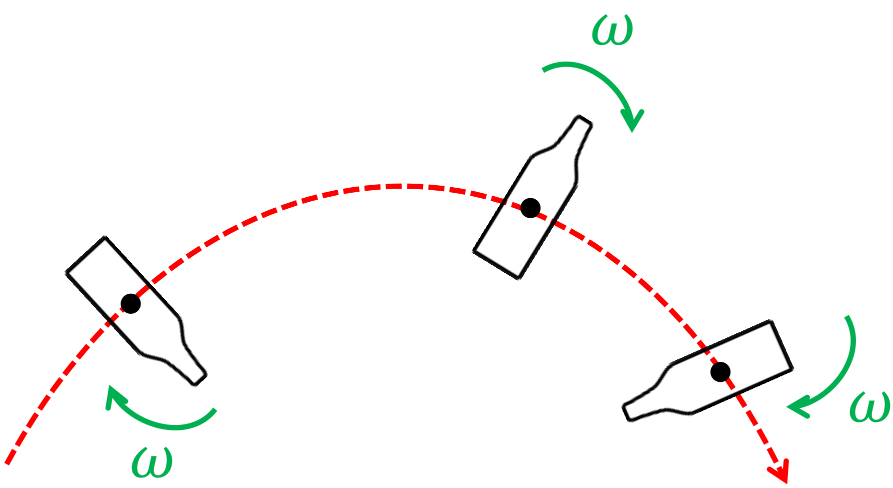



Figure 12




$$\vec{F}^e = M\frac{d^2\vec{R}}{dt^2}$$

Figure 13


Figure 14a

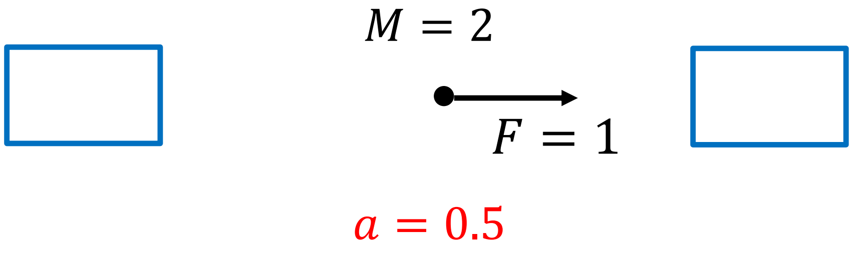



Figure 14b

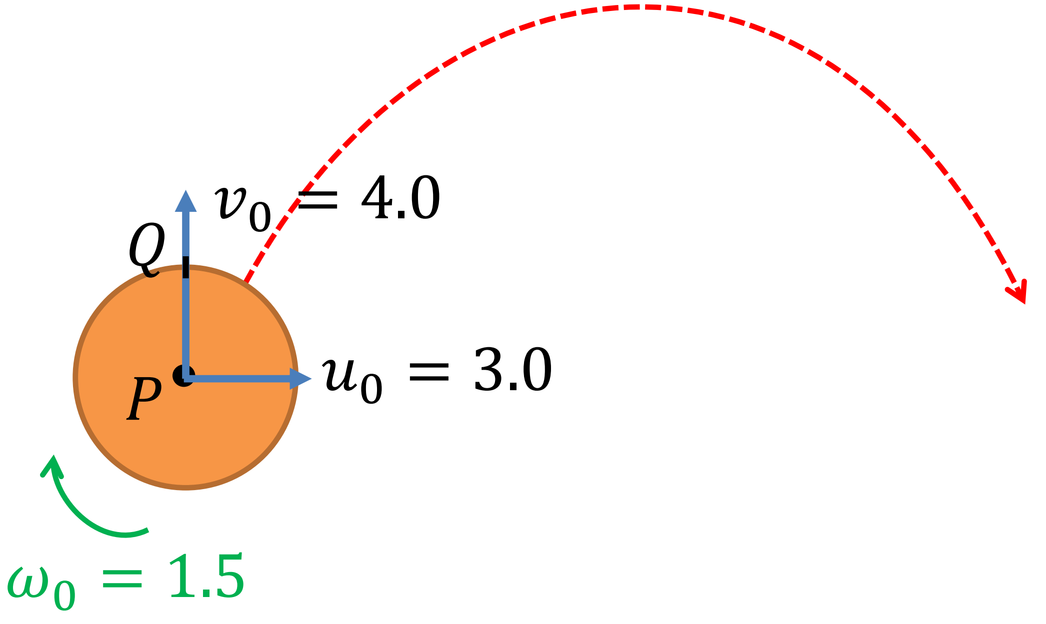



Figure 15