Line integrals

October 3, 2015

The main topic is the integral of a vector function along a path (such as the work done by a force): $\int \vec{F}(\vec{r}) \cdot d\vec{r}$*. Such integrals are defined, and ways to evaluate them are discussed: numerical methods and reduction to ordinary integrals. The opportunity is also taken to discuss other integrals over a curve, e.g.,* $\int F(\vec{r})\, ds$*, where* s *is the path length. To be self-contained, some of the material on work and energy is repeated in a more general language.*

Contents

1 Definition of line integral and numerical evaluation

1.1 Definition

A line integral

$$I = \int_\gamma \vec{F} \cdot \vec{r} \tag{1}$$

is defined for a vector function $\vec{F}$ which depends on position (i.e., a vector field):

$$\begin{aligned} & \vec{F} = \vec{F}(\vec{r}) \\ = \; & F_x(x,y,z)\,\mathbf{i} + F_y(x,y,z)\,\mathbf{j} + F_z(x,y,z)\,\mathbf{k} \end{aligned} \tag{2}$$

and a path $\gamma = \gamma(A, B)$ which is a curve with a sense of direction (i.e., going from A to B rather than the other way round). The representation in Cartesian coordinates can be generalized to n dimensions, in an obvious manner. Most of the examples will be in 2D.

The integral is defined as

$$\int_\gamma \vec{F} \cdot \vec{r} \;=\; \lim \sum \vec{F}(\vec{r}) \cdot \Delta\vec{r} \tag{3}$$

which is a shorthand for the following procedure (**Figure 1**):

- Chop the path γ into short segments $\Delta\vec{r}$, each of which can be regarded as a vector along the path, in the direction specified by the sense of γ.

- Evaluate $\vec{F}$ at some point (ideally the midpoint) in the segment. Calculate the dot product

$$\vec{F} \cdot \Delta\vec{r} = F_x \Delta x + F_y \Delta y + F_z \Delta z \tag{4}$$

- Add up all the contributions.

- Repeat this procedure for shorter and shorter segments and take the limit $|\Delta\vec{r}| \to 0$.

In practice, a good answer will be obtained for some small but finite $\Delta\vec{r}$, and we often omit the symbol lim in (3).

1.2 Motivation

Such line integrals arise in elementary physics in at least three contexts.

- $\vec{F}$ is the force, and the integral is the work done.
- The line integral of the electric field $\vec{E}$ around a closed loop is the EMF. Faraday's law relates EMF to the rate of change of magnetic flux.
- In time-independent situations, the line integral of the magnetic field $\vec{B}$ around a closed loop γ is related to the current going through the surface S bounded by γ; this is Ampere's law for magnetostatics.

1.3 Additivity

The following additivity properties are obvious.

Additivity in the integrand

$$\int_\gamma \left(\sum_i \alpha_i \vec{F}_i \right) \cdot d\vec{r} \quad = \quad \sum_i \alpha_i \int_\gamma \vec{F}_i \cdot d\vec{r} \qquad (5)$$

where $\vec{F}_1, \vec{F}_2, \ldots$ are several vector fields. Note that the sum on the LHS is a vector sum, whereas the sum on the RHS is a scalar sum.

Additivity in the path
The additivity in the path arises because of the additivity in $\Delta\vec{r}$ and can be expressed in an obvious shorthand as

$$\int_{\gamma_1} + \int_{\gamma_2} \quad = \quad \int_\gamma \qquad (6)$$

where $\gamma_1 = \gamma_1(A, B)$, $\gamma_2 = \gamma_2(B, C)$, and

$$\gamma = \gamma(A, C) = \gamma_1(A, B) + \gamma_2(B, c) \qquad (7)$$

is the composite path joining A and C (**Figure 2**). Note that γ has to be the composite path, not any other path joining A and C.

Doubling back
If $\gamma_2 = -\gamma_1$, i.e., the same path traversed in the opposite direction, then the integrals will cancel, because on the two paths, $\vec{F}$ are the same but $\Delta\vec{r}$ are opposite.

1.4 Examples

The definition is illustrated by several examples, which are basically the same as those used in the module on work. The numerical schemes can be found in the spreadsheet line.xlsx. Consider two force fields $\vec{F}$:

$$\begin{aligned} \vec{F}_1(x, y) &= 3x^2\,\mathbf{i} + y^2\,\mathbf{j} \\ \vec{F}_2(x, y) &= 3y^2\,\mathbf{i} + x^2\,\mathbf{j} \end{aligned} \qquad (8)$$

and two paths linking the same endpoints A and B (see **Figure 3**):

$$\begin{aligned} \gamma_1 &= \text{arc of radius } R \text{ from angle } \theta_A \text{ to } \theta_B \\ \gamma_2 &= \text{straight line linking the points} \\ &\quad\ A \text{ and } B \end{aligned} \qquad (9)$$

where

$$\begin{aligned} &R = 2 \quad , \quad \theta_A = 30^\text{o} \quad , \quad \theta_B = 60^\text{o} \\ &x_i = R\cos\theta_i \quad , \quad y_i = R\sin\theta_i \end{aligned} \qquad (10)$$

$i = A, B$, so that specifically $(x_A, y_A) = (\sqrt{3}, 1)$, $(x_B, y_B) = (1, \sqrt{3})$.

Example 1
Calculate the line integral of $\vec{F}_1$ along γ_1. This is the same as an Example in the module on work.

Divide the path into *small* intervals between the points $\theta = 30, 31, \ldots, 59, 60$ (all angles in degrees); each interval is a short vector $\Delta\vec{r}$. Calculate (4) for each interval (but here without the z-component). We choose arbitrarily to evaluate $\vec{F}$ at the *beginning* of each interval. The calculation can be handled conveniently using a spreadsheet (sheet1), giving $I = -2.8952$. A slightly better calculation, by evaluating $\vec{F}$ at the *midpoint* of each interval (sheet 2), gives $W = -2.7974$. The exact answer (see below) is $(2 - 6\sqrt{3})/3 = -2.7974$.

Note that in the spreadsheet we skip the last entry: The row for $\theta = 59$ gives the integral for the interval $\theta = 59$ to 60. §

Problem 1
Using the same spreadsheet as a template, calculate the line integral of $\vec{F}_2$ along γ_1. §

Example 2
Calculate the work done by $\vec{F}_1$ along γ_2.

A general point along the path is

$$\begin{aligned} x(t) &= (1-t)x_A + tx_B \\ y(t) &= (1-t)y_A + ty_B \end{aligned} \qquad (11)$$

$0 \leq t \leq 1$. Take short intervals between the points with say $t = 0, 0.02, 0.04, \ldots 0.98, 1.00$, with $\vec{F}$ evaluated at the midpoint of each interval. The answer is $I = -2.7974$ (sheet 3), exactly as in Example 1. In this case, the integral is the same for the two paths. (We cannot yet say that it is independent of path, since that would require us to check *every* path.) §

Problem 2
Using the same spreadsheet as a template, calculate the line integral for $\vec{F}_2$ along γ_2. It will be found that in this case the integral is dependent on path. §

2 Reduction to ordinary integrals

Consider one term in (4) and take the limit. We would have an integral of the form

$$\int_\gamma F_x(x, y, z)\, dx \qquad (12)$$

When we integrate over dx, y and z cannot be regarded as "other" (i.e., independent) variables, because on a given path the Cartesian coordinates are related. We must reduce everything to *only one* variable, by parametrization of a path.

2.1 Parametrization of path

Path
Formally, a *path* γ is a continuous set of points

$$\vec{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} \qquad (13)$$

where $x(t)$, $y(t)$, $z(t)$ are functions of a *path parameter* t, ranging from t_A to t_B. The sense of the path is defined conventionally by increasing t.

Example 3
If a projectile is thrown at an angle, then its path is, in obvious notation,

$$\begin{aligned} x(t) &= u_0 t \\ y(t) &= v_0 t - (1/2) g t^2 \end{aligned} \qquad (14)$$

where t is the time, say over the interval $0 \leq t \leq T = 2v_0/g$.

The path parameter need not be the time, and indeed can be arbitrarily transformed. For example, we can use the path parameter $s = t^2$ and

$$\begin{aligned} x(t) &= u_0 s^2 \\ y(t) &= v_0 s^2 - (1/2) g s^4 \end{aligned} \qquad (15)$$

The line integral etc. are all invariant under such relabelling of the path parameter. §

Example 4
The path γ_1 can be described by the path parameter θ:

$$x(\theta) = R\cos\theta \qquad , \qquad y(\theta) = R\sin\theta \qquad (16)$$

Except for the path parameter, all other variables on the RHS (e.g., R) must be constants. §

Example 5
The path γ_2 is described by (11) in terms of the interpolating parameter t, with $0 \leq t \leq 1$. §

Example 6
As an example in 3D, consider:

$$\begin{aligned} x(\theta) &= R\cos\theta \\ y(\theta) &= R\sin\theta \\ z(\theta) &= a\theta \end{aligned} \qquad (17)$$

where say $0 \leq \theta \leq 6\pi$. This describes a spiral with 3 turns and a *pitch* of $p = 2\pi a$. §

Tangent to the path
A tiny displacement $\Delta\vec{r}$ along the path is tangent to the path. Since it is inconvenient to deal with infinitesimal quantities, let us divide by Δt where t is the path parameter. Thus define

$$\vec{V} = \frac{d\vec{r}}{dt} \qquad (18)$$

which is a vector tangent to the path. A unit tangent is obtained by

$$\vec{v} = \frac{\vec{V}}{|\vec{V}|} \qquad (19)$$

If t is the time, then $\vec{V}$ is the velocity.

Example 7
Find the tangent vector to γ_1 at an arbitrary angle θ.

From the parametrization (16),

$$
\begin{aligned}
V_x &= dx/d\theta = -R\sin\theta \\
V_y &= dy/d\theta = R\cos\theta \\
|V| &= R \\
v_x &= -\sin\theta \\
v_y &= \cos\theta
\end{aligned} \tag{20}
$$

Sketch this vector and understand the signs. §

2.2 Integration over path parameter

In the definition of the line integral, e.g. (1), divide and multiply by dt (or in the discrete version by Δt):

$$
\begin{aligned}
\int_\gamma \vec{F}\cdot d\vec{r} &= \int_{t_A}^{t_B} \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt \\
&= \int_{t_A}^{t_B} \left(F_x\frac{dx}{dt} + F_y\frac{dy}{dt} + F_z\frac{dz}{dt}\right) dt \\
&\equiv \int_{t_A}^{t_B} P(t)\,dt
\end{aligned} \tag{21}
$$

which is now expressed as an ordinary integral, with the integrand $P(t)$ being the expression in brackets. If $\vec{F}$ is the force and t is the time, then $P(t)$ is just the power $\vec{F}\cdot\vec{V}$, whose integral over time is the total work done.

Once the path has been parameterized, line integrals are reduced to ordinary integrals — and all the techniques for dealing with the latter can be employed.

2.3 Examples

Example 8
Repeat Example 1 using a path parameter.

$$
\begin{aligned}
x &= R\cos\theta \\
y &= R\sin\theta \\
V_x &= dx/d\theta = -R\sin\theta \\
V_y &= dy/d\theta = R\cos\theta \\
F_x &= 3x^2 = 3R^2\cos^2\theta \\
F_y &= y^2 = R^2\sin^2\theta
\end{aligned} \tag{22}
$$

The integrand is

$$
\begin{aligned}
& P(\theta) = F_xV_x + F_yV_y \\
&= (3R^2\cos^2\theta)(-R\sin\theta) + (R^2\sin^2\theta)(R\cos\theta) \\
&= R^3(-3\cos^2\theta\sin\theta + \sin^2\theta\cos\theta)
\end{aligned} \tag{23}
$$

It is left as an exercise to evaluate $\int P(\theta)\,d\theta$. §

Problem 3
Complete the final step in the last example, for the endpoints being those in Example 1. §

Problem 4
In a similar manner calculate the line integral for F_2 along the same path. §

Problem 5
In a similar manner (now using the parameter t), calculate the line integral of $\vec{F}_1$ and $\vec{F}_2$ along γ_2. §

3 Other integrals along a line*

** This part is not needed for dealing with work and energy, and can be skipped.*

In the line integrals defined above, the infinitesimal element is $d\vec{r}$, a *vector*. The contribution of each segment depends on the direction of that segment. Other situations arise in which the infinitesimal element is ds, the length of the segment, which is a *scalar*. In this case, if the path doubles back on itself, the two segments do *not* cancel: the integrand is the same, and ds is also the same.

3.1 The path length

The commonest integral is the length of a path. A small segment of path has a length

$$
\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2} \tag{24}
$$

and upon taking the continuum limit, the total length is given formally by

$$
s = \int ds = \int \sqrt{(dx)^2 + (dy)^2} \tag{25}
$$

It may seem perplexing to have the differentials appearing in this manner. Divide and multiply by dt (and remember that inside the square root we need two powers of dt):

$$
\begin{aligned}
s &= \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&\equiv \int g(t)\,dt
\end{aligned} \tag{26}
$$

reduced to an ordinary integral.

Example 9
Find the length of the path in (14) from $t = 0$ to $t = T \equiv 2v_0/g$. Take $u_0 = v_0 = 5$, $g = 10$.

We have $T = 1$ and

$$\begin{aligned} dx/dt &= 5 \\ dy/dt &= 5\,(1-2t) \\ s &= 5\int_0^1 \sqrt{1+(1-2t)^2}\,dt \end{aligned} \tag{27}$$

This ordinary integral can be handled in various ways. §

Problem 6
Evaluate (27)
(a) numerically using a spreadsheet, and
(b) analytically.
Hint: Put $\tau = 2t - 1$ and then $\tau = \sinh\theta$. §

Problem 7
An ellipse with semi-major axis a and semi-minor axis b is described by

$$(x/a)^2 + (y/b)^2 = 1 \tag{28}$$

We can use a parameter θ such that

$$\begin{aligned} x &= a\cos\theta \\ y &= b\sin\theta \end{aligned} \tag{29}$$

Note that θ is *not* the polar angle.
(a) Express the circumference as an integral over θ.
(b) For $a = 1$, $b = 2$, evaluate the integral numerically. (It is enough to consider one quadrant.)
(c) Try to do the integral analytically, for general a and b. Without loss of generality assume $a \le b$. §

Problem 8
Calculate the length of the spiral in Example 6. §

3.2 Another example

In the study of planetary motion, the usual starting point is to consider the sun (assumed to be so heavy that it does not move) and only one planet (say Mercury) at a distance r. This problem is readily solved, and gives Kepler orbits. But actually every *other* planet (say Uranus, mass m, at distance R from the sun) produces a small additional effect. For long-term effects, take the other planet of mass m to be "smeared out" into a ring at radius R. What is the potential Φ (PE per unit mass) experienced by the first planet at the point P? See **Figure 4**.

The mass per unit length on the ring is $\lambda = m/(2\pi R)$, and the distance between a part of the ring at X and the planet at P is $q = XP$. Thus the potential is

$$\Phi(r) = -\int \frac{G(\lambda\, ds)}{q} \tag{30}$$

which involves an integral over a line, in this case a circle. Express everything in terms of the angle θ:

$$\begin{aligned} ds &= R\,d\theta \\ q^2 &= (R\cos\theta - r)^2 + (R\sin\theta^2) \end{aligned} \tag{31}$$

Thus

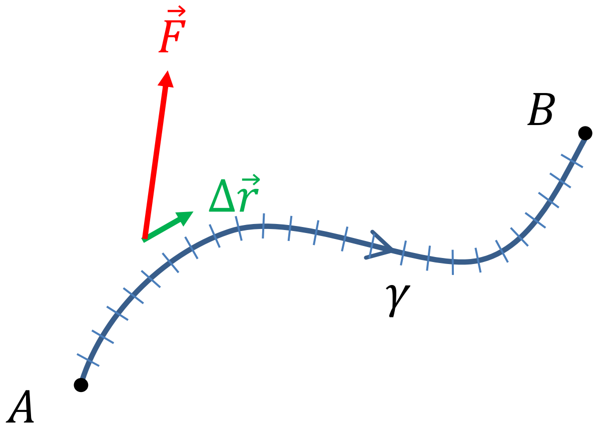
$$\begin{aligned} \Phi(r) &= -\frac{Gm}{R}\int_0^{2\pi} \frac{R\,d\theta}{\sqrt{r^2 - 2rR\cos\theta + R^2}} \\ &\equiv -\frac{Gm}{R}\, I \end{aligned} \tag{32}$$

where it is easy to see that the integral I depends only on the ratio

$$\xi = \frac{r}{R} \tag{33}$$

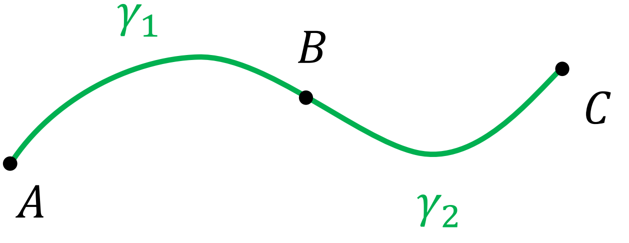
Problem 9
Evaluate I numerically for $\xi = 0.3$. This integral was also discussed in the module on *Integration: Part 1.*[1] §

[1] See KH Lo, K Young and BYP Lee, "Advance of Perihelion", Am. J. Phys., 81, 695 (2013). doi 10.1119/1.4813067.



$$\int_{\gamma} \vec{F} \cdot d\vec{r} = \lim \sum \vec{F}(\vec{r}) \cdot \Delta\vec{r}$$

Figure 1



$$\gamma(A,C) = \gamma_1(A,B) + \gamma_2(B,C)$$

Figure 2

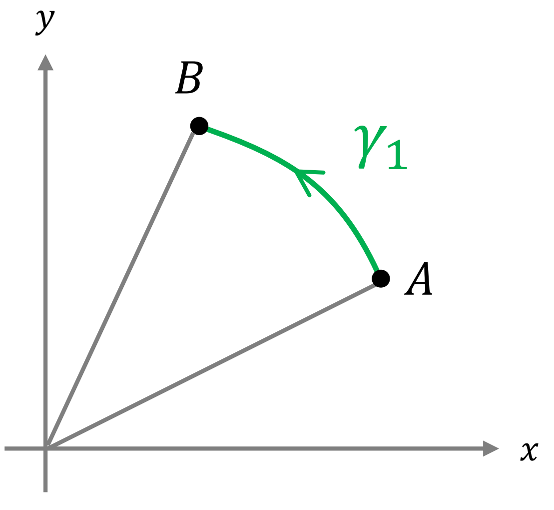


Figure 3a

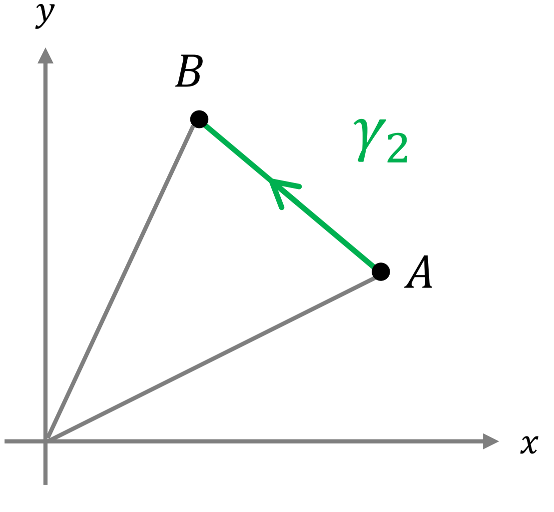


Figure 3b

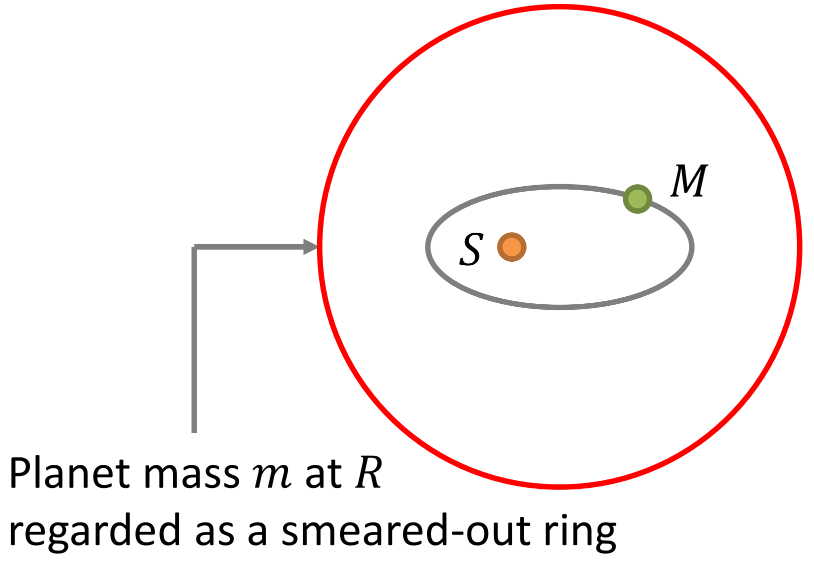


Figure 4a

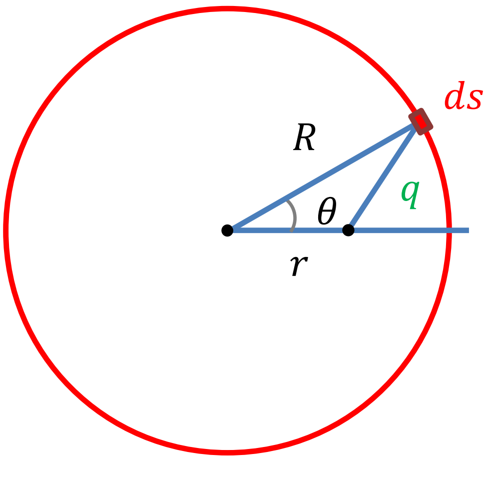


Figure 4b