Work: higher dimensions

October 2, 2015

The concept of work is introduced for n dimensions. Line integrals are defined and the condition for a force to be conservative is stated (with derivation in a later module). Students should first review the corresponding material for one dimension.

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1 Definition of work

1.1 Constant force

In terms of magnitudes
The work done by a constant force $\vec{F}$ in moving an object through a displacement $\vec{s}$ is defined as

$$\begin{aligned} W &= \vec{F}\cdot\vec{s} \\ &= F_{\parallel}\, s = Fs\cos\alpha \end{aligned} \quad (1)$$

where $F_{\parallel}$ is the *parallel component* of $\vec{F}$, i.e., the component along the displacement, and α is the angle between $\vec{F}$ and $\vec{s}$ (**Figure 1**).

Example 1
A mass m is subject to gravitational force mg. Find the work done by an external force $\vec{F}_e$ in moving the mass without acceleration (a) vertically upwards by a height h from A to B; and (b) at 45° to the vertical, from A to C at the same height as B (so that the distance travelled is $\sqrt{2}\,h$); see **Figure 2**.

Choose coordinates such that x is horizontal and $+y$ points vertically upwards.

$$\vec{F} = -\,mg\,\mathbf{j} \quad (2)$$

(a) For the first case,

$$\begin{aligned} \vec{F}_e &= -\vec{F} = mg\,\mathbf{j} \\ \vec{s} &= h\,\mathbf{j} \\ W &= \vec{F}_e\cdot\vec{s} = mgh \end{aligned} \quad (3)$$

(b) For the second case,

$$\begin{aligned} W &= F_e s\cos\alpha \\ &= (mg)(\sqrt{2}\,h)\cos 45^{\circ} \\ &= mgh \end{aligned} \quad (4)$$

Alternately we can evaluate the dot product using Cartesian components:

$$\begin{aligned} W &= \vec{F}_e\cdot\vec{s} \\ &= F_{ex}s_x + F_{ey}s_y \\ &= (0)(h) + (mg)(h) = mgh \end{aligned} \quad (5)$$

The two methods of course agree. §

In terms of Cartesian components
The displacement is the same as change of the position $\vec{r}$:

$$\begin{aligned} \vec{s} &= \Delta\vec{r} \\ (s_x, s_y, s_z) &= (\Delta x, \Delta y, \Delta z) \end{aligned} \quad (6)$$

It is obvious, now generalizing to 3D and writing the work as ΔW for a small displacement:

$$\begin{aligned} W &= \vec{F}\cdot\Delta\vec{r} \\ &= F_x\Delta x + F_y\Delta y + F_z\Delta z \end{aligned} \quad (7)$$

The generalization of this formula (and others below) to n dimensions is straightforward.

Example 2
A force $\vec{F} = 9.000\,\mathbf{i} + 1.000\,\mathbf{j}$ pushes a mass from $(x, y) = (1.7321, 1.0000)$ to $(1.7143, 1.0301)$. Find the work done.

We have $\Delta x = -0.0178$, $\Delta y = 0.0301$, so applying (7),

$$\begin{aligned} W &= (9.000)(-0.0178) + (1.000)(0.0301) \\ &= -1.2937 \times 10^{-1} \end{aligned} \qquad (8)$$

Cartesian coordinates allow computation without diagrams. However, sketch the force and displacement and try to understand their directions. §

Example 3
A force which depends on position as

$$\vec{F}(x, y) = 3x^2\,\mathbf{i} + y^2\,\mathbf{j} \qquad (9)$$

pushes a mass along an arc of radius $R = 2$ from $\theta_1 = 30^{\circ}$ to $\theta_2 = 31^{\circ}$. The angles are measured from the x-axis.

The displacement is small and we can regard the motion as being along a straight line from $\vec{r}_1$ to $\vec{r}_2$, with $x_i = R\cos\theta_i$, $y_i = R\sin\theta_i$. These positions are exactly the same as those in Example 2. The force may be approximated as constant over a short interval, and if we evaluate at $\vec{r}_1$, then $F_x = 3x_1^2 = 9.000$, $F_y = y_1^2 = 1.000$, again as in Example 2. So the work done is also exactly the same. §

Problem 1
Repeat Example 3 for the displacement between $\theta_1 = 50^{\circ}$ to $\theta_2 = 51^{\circ}$. §

1.2 Linearity

Linear property
Work is linear or additive in both the force and the displacement. If there are several forces $\vec{F}_1, \vec{F}_2, \ldots$, then the work done by the net force $\vec{F} = \sum_i \vec{F}_i$ is given by $W = \sum_i W_i$, where W_i is the work done by the force F_i.[1] The same additive property will also apply to potential energy. Other consequences of linearity will be cited later.

Perpendicular component does no work
Why do we define work as in (1), in terms of a dot product, i.e., considering only the parallel component of the force? Beginning students may say: "Because the perpendicular component does no work". This answer is wrong; it is circular reasoning. The perpendicular component does no work *because* we have defined W by the dot product.

Suppose we define "work" as $W' = F\,s$, i.e., the product of the magnitudes. Consider two forces $\vec{F}_1 = \mathbf{i}$, $\vec{F}_2 = -\mathbf{i}$ and a displacement $\vec{s} = \mathbf{j}$. If we calculate the individual values, $W_1' = 1$, $W_2' = 1$. But the total force is $\vec{F} = 0$, so using it to calculate the total "work" would give $W' = 0 \neq W_1' + W_2'$. You can of course define "work" this way, but the resultant concept would not be of much use.

1.3 Variable force

Continue with Example 3 for a larger interval. It is no longer legitimate to regard the force as constant. The numerical calculations are all in the spreadsheet work-b.xls.

Example 4
As in Example 3, find the work done in moving the mass along the arc, from $\theta_A = 30^{\circ}$ to $\theta_B = 60^{\circ}$. In the following, all angles are expressed in degrees.

Repeat Example 3 for each *small* interval between the points $\theta = 30, 31, \ldots, 59, 60$. The contribution for the first interval is just Example 3; the contribution for the interval $\theta = 50$ to 51 is just Problem 1. The calculation can be handled conveniently using a spreadsheet (sheet 1), giving $W = -2.8952$. A slightly better calculation, by evaluating $\vec{F}$ at the middle of each interval (sheet 2) gives $W = -2.7974$. The exact answer (if you know calculus) is $(2 - 6\sqrt{3})/3 = -2.7974$.

Note that in the spreadsheet we skip the last entry for the work: The row for $\theta = 59$ gives the work done for the interval $\theta = 59$ to 60. §

This example leads to the following rule.

- Chop the whole path γ from A to B into short intervals $\Delta\vec{r}$.
- Calculate the work done in each interval by (7), where $\vec{F}$ is evaluated anywhere in the short interval (but optimally in the middle).
- Add it all up.
- Theoretically, repeat the calculation for smaller and smaller $\Delta\vec{r}$ (which would involve

[1] Note: The first sum here involves vector addition and the second sum involves scalar addition.

more and more terms) until the answer converges.

All this is summarized by the shorthand

$$W(\gamma) \quad = \quad \lim_{\Delta\vec{r}\to 0} \sum \vec{F}\cdot\Delta\vec{r} \qquad (10)$$

or less formally (with the limit understood) as

$$\boxed{W(\gamma) = \sum \vec{F}\cdot\Delta\vec{r}} \qquad (11)$$

In practice, this means using a finite but sufficiently small $\Delta\vec{r}$. Note that W is labelled not just by the endpoints but by the path.

In general, "chop and add" is the *only* way of calculating the work done.

Example 5
Calculate the work done by the same force as in Example 4, for the same endpoints, but along a straight line joining them.

The endpoints are $(x_A, y_A) = (\sqrt{3}, 1)$ and $(x_B, y_B) = (1, \sqrt{3})$, and a general point along the path:

$$\begin{aligned} x(t) &= (1{-}t)x_A + tx_B \\ y(t) &= (1{-}t)y_A + ty_B \end{aligned} \qquad (12)$$

with t $(0 \le t \le 1)$. Take short intervals between the points with say $t = 0, 0.02, 0.04, \ldots 0.98, 1.00$, and evaluate the force at the middle of each interval (sheet 3). The answer is $W = -2.7974$, exactly as in Example 4. In this case, the work done is the same for both paths. §

Problem 2
Use spreadsheets to calculate the work done along the two paths in Example 4 and Example 5, for the force

$$\vec{F}(x, y) \quad = \quad 3y^2\,\mathbf{i} + x^2\,\mathbf{j} \qquad (13)$$

In this case that the work done along the two paths are different. §

Linearity again
This limiting procedure only makes sense if W is linear in $\Delta\vec{r}$. Suppose the basic definition is quadratic in the size of the interval. Then if each interval is further chopped into two, there would be 2 times as many terms, each 1/4 the original value, so the sum is reduced by 1/2 — the limit would not exist.

Zitterbewegung*
(**This advanced diversion can be skipped.*)
Consider a range L so small that the force can be considered constant, and suppose a particle moves from A to B (**Figure 3**). The displacement is

$$\Delta\vec{r} \quad = \quad \vec{r}_B - \vec{r}_A \qquad (14)$$

as shown by the thick arrow in **Figure 3**. The work done is $W = \vec{F}\cdot\Delta\vec{r}$.

But suppose the particle does not go along a straight line, but has a "trembling motion" (in German "Zitterbewegung") with little zigzag segments $\Delta\vec{r}_1, \Delta\vec{r}_2, \ldots$, as shown by the thin lines in **Figure 3**. Then the individual pieces of work done would be

$$W_i \quad = \quad \vec{F}\cdot\Delta\vec{r}_i \qquad (15)$$

These add up to $\vec{F}\cdot\Delta\vec{r}$, as if the particle had gone along the straight path — because W is additive in the displacement. So we do not need to know the fine resolution of the path below the scale L. Without linearity, we would have to know about any "trembling motion" of the particle.

You would object: By Newton's law, the particle cannot zigzag. But who says Newton is right? At sufficiently small length scales, Newtonian mechanics breaks down, there is uncertainty in the position, and in some sense (most precisely in the sense of the Feynman path integral) the particle can go along any possible path. Fortunately, because of the above argument, we do not need to worry.

This point is mentioned to emphasize that the concepts of work and potential energy go beyond Newtonian mechanics. Quantum mechanics involves the potential energy in an essential way. Linearity allows us to define work and hence potential energy independent of the fine details of the path.

1.4 Line integral

The limiting value of a sum such as (11) is called a *line integral*. The work is defined for a *path* γ with endpoints A and B:

$$W(\gamma(A, B)) \quad = \quad \int_{\gamma(A,B)} \vec{F}(\vec{r})\cdot d\vec{r} \qquad (16)$$

Consider a fixed starting point O (not necessarily the origin) and allow the endpoint to be a variable, say $\vec{r}$.

$$W(\gamma(O,\vec{r})) = \int_{\gamma(O,\vec{r})} F(\vec{r}\,') \cdot d\vec{r}\,' \qquad (17)$$

Here, $\vec{r}$ is the endpoint or upper limit, with a definite value, while in the integrand $\vec{r}\,'$ is a dummy variable along the path. But when there is no danger of confusion, we can write in a slightly sloppy way:

$$W(\gamma(O,\vec{r})) = \int_{\gamma(O,\vec{r})} \vec{F}(\vec{r}) \cdot d\vec{r} \qquad (18)$$

Line integrals are discussed systematically in a later module. Here we only need two properties:

- The definition in terms of the limit of a sum, and the corresponding numerical method (the simplest version).
- The fundamental theorem of calculus: integration is the reverse of differentiation. This will be discussed formally in a later module, but more physically here and in the next module in terms of the relationship between potential energy and force.

2 Reducing to ordinary integral

In this Section, we briefly sketch how a line integral can converted to an ordinary integral. A more formal discussion is presented in a later module.

Take (11) and write out the dot product in Cartesian coordinates:

$$W = \sum (F_x\,\Delta x + F_y \Delta y + F_z\,\Delta z) \qquad (19)$$

Passing to the limit of small intervals and writing out explicitly the dependence of the force fields on position:

$$W = \int_\gamma F_x(x,y,z)\,dx + F_y(x,y,z)\,dy + F_z(x,y,a)\,dz \qquad (20)$$

where we have indicated the integration as over a path γ. How do we do such an integral?

When you do dx, you cannot regard y and z as simply "other" variables. If the path is a part of a circle of radius R in the x–y plane, then y is just a shorthand for $\sqrt{R^2 - x^2}$. Thus the key is to *reduce everything to one single independent variable.* We illustrate with one example.

Example 6
Redo Example 3 by converting to an ordinary integral.

Obviously the angle θ is the most convenient independent variable. So we have

$$\begin{aligned} x &= R\cos\theta \\ y &= R\sin\theta \\ dx &= -R\sin\theta\,d\theta \\ dy &= R\cos\theta\,d\theta \\ F_x &= 3x^2 = R^2(3\cos^2\theta) \\ F_y &= y^2 = R^2(\sin^2\theta) \end{aligned} \qquad (21)$$

If we put all this into (20) we would get an expression of the form

$$W = \int G(\theta)\,d\theta \qquad (22)$$

between the limiting values θ_A and θ_B.

Problem 3
Finish the above calculation and verify the answer $(2 - 6\sqrt{3})/3$ cited in Example 4. §

Problem 4
Find the work done in Example 5 analytically, using the variable t as the independent variable. Verify the answer $(2 - 6\sqrt{3})/3$ cited in Example 4. §

Problem 5
Repeat Problem 2 analytically, for the two paths. §

The method presented here allows a line integral to be converted to an ordinary integral, but does not guarantee that the latter can be evaluated analytically (even though we can do so in the simple examples above). But it is still often useful to convert to an ordinary integral before resorting to numerical methods.

3 Dependence on path?

The concept of potential energy (PE), and hence of the conservation of energy, can only apply to *conservative forces*, i.e., those for which the work done is independent of path. In the discussion for 1D, a necessary condition was pointed out: the force depends only on position, $\vec{F} = \vec{F}(\vec{r})$, thus excluding for example magnetic force and friction. But beyond 1D, there are additional conditions.

3.1 Reduction to closed loops

Figure 4a shows two paths γ_1 and γ_2, both going from A to B. The question we want to ask is whether

$$\begin{aligned} W(\gamma_1) &\stackrel{?}{=} W(\gamma_2) \\ \int_{\gamma_1} \vec{F} \cdot d\vec{r} &\stackrel{?}{=} \int_{\gamma_2} \vec{F} \cdot d\vec{r} \end{aligned} \tag{23}$$

Define the *closed path* γ as (**Figure 4b**)

$$\gamma = \gamma_1 - \gamma_2 \tag{24}$$

in other words, go along γ_1 and then in the reverse direction along γ_2. Then (23) is the same as asking whether

$$\begin{aligned} W(\gamma) &\stackrel{?}{=} 0 \\ \oint_{\gamma} \vec{F} \cdot d\vec{r} &\stackrel{?}{=} 0 \end{aligned} \tag{25}$$

The little circle on the integral sign denotes a line integral over a *closed* path. Since we need (23) to hold for any pair of paths γ_1 and γ_2 that share the same endpoints, the condition (25) must hold for *every* closed loop.

3.2 Examples

Just to gain some familiarity with the concepts, let us evaluate some closed loop integrals.

Example 7
Take the closed loop to be a circle of radius R and the force to be

$$\begin{aligned} \vec{F} &= x\,\mathbf{i} + y\,\mathbf{j} \\ &= R\,(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) \end{aligned} \tag{26}$$

The line element is

$$\begin{aligned} d\vec{r} &= dx\,\mathbf{i} + dy\,\mathbf{j} \\ &= d(R\cos\theta)\,\mathbf{i} + d(R\sin\theta)\,\mathbf{j} \\ &= R\,(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})\;d\theta \end{aligned} \tag{27}$$

and the integral is

$$\begin{aligned} W &= \int_0^{2\pi} \vec{F} \cdot d\vec{r} \\ &= \int_0^{2\pi} R^2(-\sin\theta\cos\theta + \sin\theta\cos\theta)\,d\theta \\ &= 0 \end{aligned} \tag{28}$$

We do not know yet whether the force (26) is conservative, since the present calculation does not prove that the integral is zero for *every* closed path. §

Example 8
For the same path, calculate the work done for

$$\begin{aligned} \vec{F} &= y\,\mathbf{i} - x\,\mathbf{j} \\ &= R\,(\sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j}) \end{aligned} \tag{29}$$

The expression for $d\vec{r}$ is the same as in (27) and the integral is

$$\begin{aligned} W &= \int_0^{2\pi} [R\,(\sin\theta\,\mathbf{i} - \cos\theta\,\mathbf{j})] \\ &\qquad \cdot [R\,(-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j})\;d\theta] \\ &= R^2 \int_0^{2\pi} \left(-\sin^2\theta - \cos^2\theta\right)\,d\theta \\ &= -2\pi R^2 \neq 0 \end{aligned} \tag{30}$$

So the force (29) is not conservative. §

Problem 6
Sketch the force vectors at the positions $(x, y) = (1,0), (0,1), (-1,0), (0,-1)$ for both Example 7 and Example 8, and give a qualitative explanation as to why one closed loop integral is zero and the other is not. §

3.3 Condition for conservative force

To show that a force field is not conservative, it is enough (as in Example 8) to find *one* closed loop for which the work done is nonzero. But to prove that the force field is conservative would require us to check *every* closed loop; this is not possible. So a better method is needed.

Here we only state the result and sketch the steps of the proof; details are in the modules that discuss line integrals more formally and mathematically. The argument goes in two steps.

First, every closed loop (**Figure 5a**) can be regarded as the sum of small rectangular loops (**Figure 5b**); the overlapping internal lines cancel. So it suffices to check every small rectangular loop.

Second, consider a small rectangle, say in the x–y (**Figure 6**). Here we sketch only the main ideas.

- For the two horizontal segments, only F_x is involved. They are evaluated at different values of y, and enter with opposite signs because the segments are traversed in opposite directions. Their difference is related to $\partial F_x/\partial y$.
- By the same arguments, the two vertical segments give a result related to $\partial F_y/\partial x$.

Putting these together and generalizing to other planes, the condition for the closed loop integral to be zero is (details in a later module)

$$\boxed{(\mathrm{curl}\,\vec{F})_{ij} = 0} \tag{31}$$

for any Cartesian indices i, j, and curl $\vec{F}$ is, e.g.,

$$(\mathrm{curl}\,\vec{F})_{xy} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \tag{32}$$

The object curl $\vec{F}$ is antisymmetric in its indices. In n dimensional space it has $n(n-1)/2$ independent components (respectively $0, 1, 3, 6, 10, \ldots$ components for $n = 1, 2, 3, 4, 5, \ldots$). In 1D, it has no nontrivial component and the condition (31) is vacuous. This slightly unfamiliar notation makes the above concepts valid in any number of dimensions.

Notation for 3D

In 3D (and only in 3D), it is possible to adopt a simpler notation using only one index: define a vector Ω such that

$$\Omega_z = (\mathrm{curl}\,\vec{F})_{xy} \tag{33}$$

etc., or more formally

$$\Omega_i = \frac{1}{2}\epsilon_{ijk}\,(\mathrm{curl}\,\vec{F})_{jk} \tag{34}$$

where ϵ_{ijk} is the totally antisymmetric Levi-Civita symbol and the summation convention is adopted. The factor $1/2$ appears because the terms with jk and kj are double counted. Obviously we can also write $\vec{\Omega}$ as a cross product:

$$\vec{\Omega} = \vec{\nabla} \times \vec{F} \tag{35}$$

and the condition of conservative force is

$$\boxed{\vec{\nabla} \times \vec{F} = 0} \tag{36}$$

This notation is only possible in 3D. For example, in (33), the indices on the RHS are x, y and the index on the LHS is "the remaining index". There is no unique remaining index in say 4D.

Problem 7

Go back to Example 7 and calculate curl $\vec{F}$. Thus show that the work done is zero around *any* closed loop. §

Problem 8

Go back to Example 8. Calculate curl $\vec{F}$. Thus show that the work done around a closed loop is in general nonzero. §

Problem 9

This is to set the stage for the next Problem. Show that

$$\frac{\partial r}{\partial x} = \frac{x}{r} \tag{37}$$

Hint:

$$\frac{\partial r}{\partial x} = \frac{\partial r^2}{\partial x} \Bigg/ \frac{dr^2}{dr} \tag{38}$$

This formula is useful in many contexts and should be remembered. §

Problem 10

Show that any central force, i.e.,

$$\begin{aligned} \vec{F} &= f(r)\,\mathbf{e}_r = f(r)\,\frac{\vec{r}}{r} \\ F_x &= f(r)\,\frac{x}{r} \end{aligned} \tag{39}$$

is conservative. §

Problem 11

Suppose a force field is given in terms of a scalar function (see next module) as

$$F_x(x, y, z) = -\frac{\partial U(x, y, z)}{\partial x} \tag{40}$$

etc. Show that such a force field satisfies (31). §

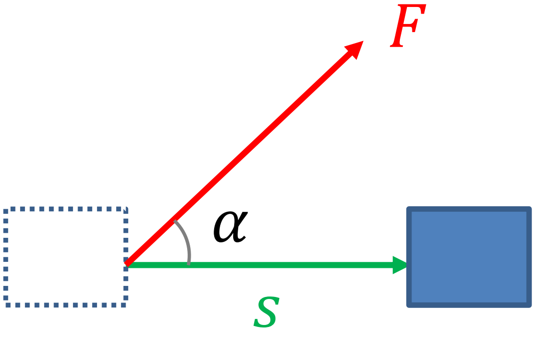

Figure 1

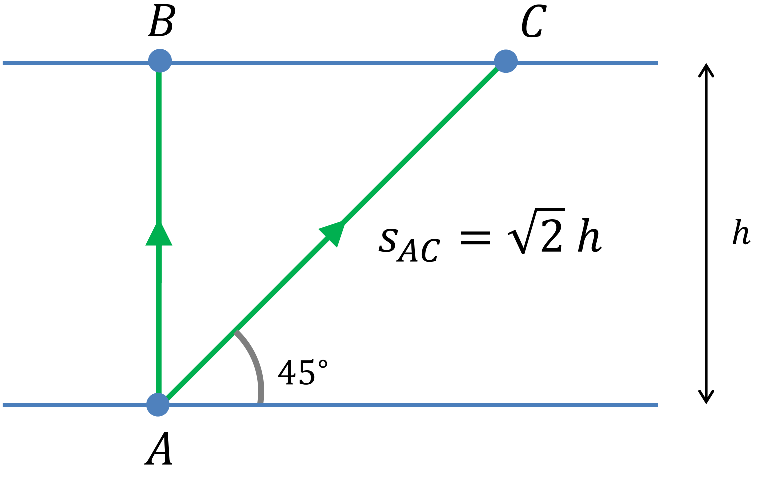

Figure 2

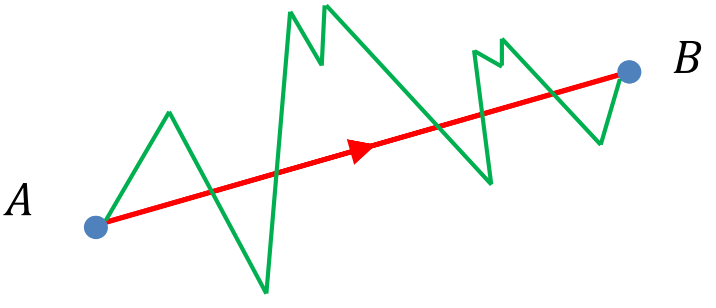

Figure 3

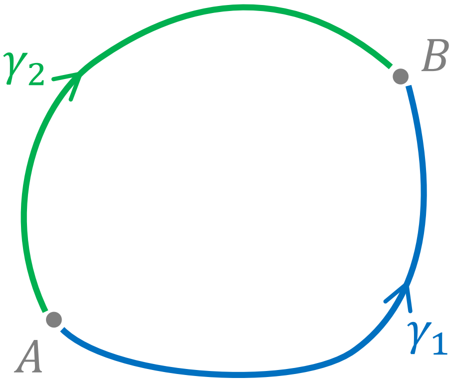

Figure 4a

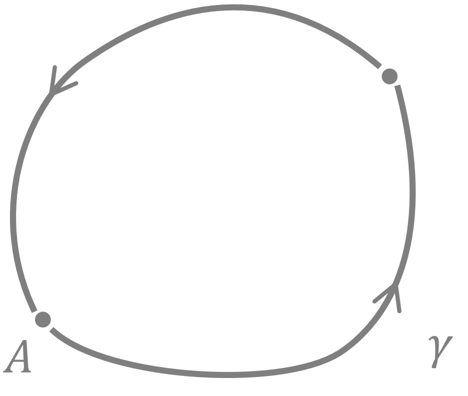

Figure 4b

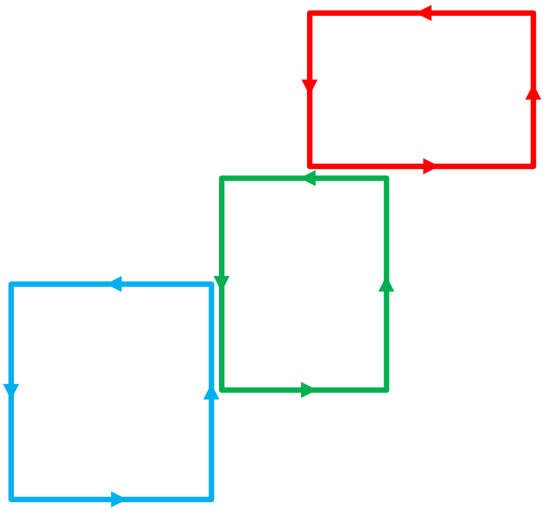

Figure 5a

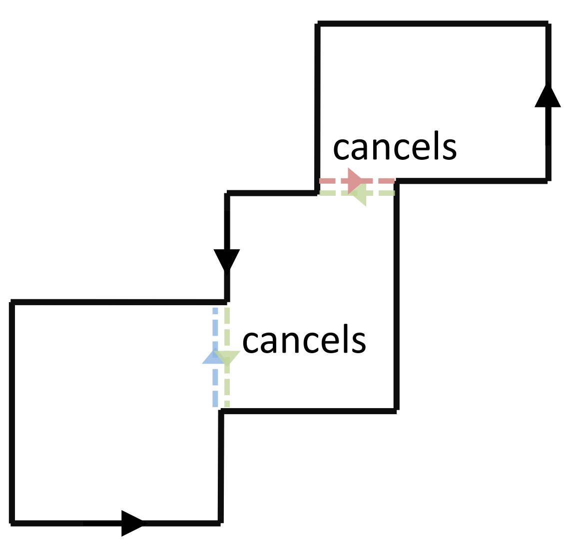

Figure 5b

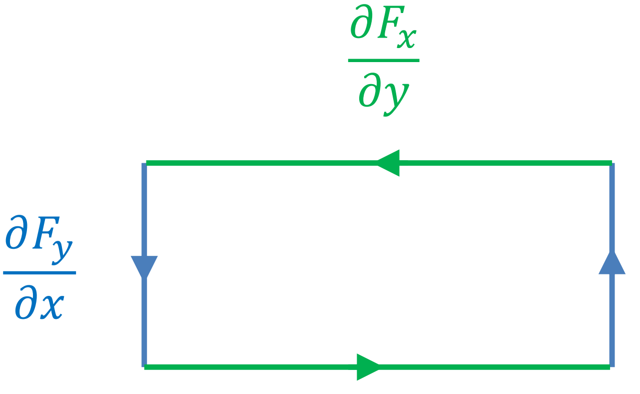

Figure 6