Vectors

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Vectors are introduced both physically and also algebraically. The dot product and the cross product are introduced, and their properties derived.

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1 Physical introduction

Vectors are introduced from a physical point of view — which, like learning arithmetic in childhood by playing with toy blocks, is not entirely logical, nor satisfactory to mathematicians, but pedagogically useful.

1.1 The prototype vector and naive definition

Prototype vector

The displacement from one point A to another point B, i.e., the *directed* or *signed* line segment AB is the prototype vector (**Figure 1**). Thus, the naive definition of a vector is *a quantity having direction as well as magnitude.*[1]

Notation

Vectors are indicated as, for example, $\vec{a}$ or $\mathbf{a}$.[2] Bold letters are typically represented in handwriting by a wavy underline.

[1] This definition found in most dictionaries is really nonsense, because what is "direction"?

[2] Especially when, in relativity, 3-vectors $\mathbf{a}$ in space and 4-vectors $\vec{a}$ in spacetime have to be distinguished.

The length or *magnitude* of $\vec{a}$ is indicated in any of the following ways:

$$\text{magnitude} \;=\; |\vec{a}|, \|\vec{a}\|, \text{ or simply } a$$

1.2 Addition of vectors

If a particle moves from P to Q (displacement $\vec{a}$), and then from Q to R (displacement $\vec{b}$), the net result is to move from P to R (displacement $\vec{c}$); see **Figure 2**. Therefore, following displacements as the prototype, vectors are added by the *triangle rule* illustrated by this figure.

Subtraction is the same as addition of the negative vector (**Figure 3**)

$$\vec{a} - \vec{b} \;=\; \vec{a} + \left(-\vec{b}\right)$$

where the latter is defined as multiplication by -1 (see below), i.e., reversing the direction.

1.3 Multiplication by a scalar

If $\vec{a}$ is a vector and s is scalar, i.e., a real number, then $s\,\vec{a}$ is a vector in the same (opposite) direction if $s > 0$ ($s < 0$) but with $|s|$ times the magnitude (**Figure 4**). Division is the same as multiplication by the reciprocal.

Expected properties such as $2\,\vec{a} = \vec{a} + \vec{a}$ are obviously satisfied.

1.4 Building other vectors from the displacement

Since displacement (say $\Delta\vec{r}$) is a vector, the following, constructed by addition/ subtraction and/or multiplication by a scalar, must also be vectors.

- Division by Δt gives the velocity:

$$\vec{v} \;=\; \frac{\Delta\vec{r}}{\Delta t}$$

- Multiplication by the mass m gives the momentum:

$$\vec{p} \;=\; m\,\vec{v}$$

- The change in momentum $\Delta\vec{p}$ during a time interval Δt is also a vector, and therefore also the force:

$$\vec{F} \;=\; \frac{\Delta\vec{p}}{\Delta t}$$

By this process, different vectors are constructed, starting from a prototype.

2 Algebraic formulation

The algebraic treatment is useful for numerical accuracy, computerization, and problems in higher dimensions. It is assumed a Cartesian coordinate system is set up, with the x, y and z axes mutually perpendicular.

2.1 Components

Definition
A vector $\vec{a}$ can be specified by its projections a_x, a_y, a_z on the three axes; the case of 2D (i.e., omitting a_z) is shown in **Figure 5**. These projections are called the *components* of the vector.

We can say that a vector in 3D is the ordered triple[3]

$$\vec{a} \;=\; (a_x, a_y, a_z) \tag{1}$$

Magnitude and direction
The length or magnitude of such a vector is given by Pythagoras' theorem as

$$a = |\vec{a}| \;=\; \sqrt{a_x^2 + a_y^2 + a_z^2}$$

The direction can be specified by angles (one angle for 2D and two angles for 3D) defined in the same way as the polar coordinates for specifying a point in space.

Addition
Consider the addition of two vectors

$$\vec{c} \;=\; \vec{a} + \vec{b}$$

by the triangle rule. By projecting each of these onto one axis (say the x axis, as shown in **Figure 6**), it is obvious that the components simply add:

$$c_x \;=\; a_x + b_x \tag{2}$$

and likewise for the other components. This then provides an *algebraic* way of adding vectors, without having to draw diagrams.

[3] It is sometimes convenient to think of these triples as *column* vectors; but because columns are cumbersome to display, this alternate notation will be ignored until there is a good reason to adopt it.

Multiplication by a scalar
If the length of a vector is multiplied by a scalar s, then obviously each component is multiplied by the same factor. In other words, if $\vec{c} = s\,\vec{a}$, then

$$c_x \quad = \quad s\,a_x \quad \text{etc.}$$

So again, we have an *algebraic* way of doing scalar multiplication.

Problem 1
A person starts from a point O walks 30.0 m along the compass direction 30°, and then 40.0 m along the compass direction 60°. Find the distance and compass direction of the final position from O. §

The above problem can be solved (to some accuracy) graphically, but not the next one.

Problem 2
Warship A directs its radar in compass direction 45° and elevation 30° from the horizontal. It finds the radio pulse reflected after 40.0 μs, from an enemy plane P. The captain of warship A conveys this information to his colleague on warship B, which is 7.00 km due east (i.e., compass direction 0°). In what direction (compass direction and elevation) should warship B point its radar in order to locate P? Speed of light is 3.00×10^8 m s^{-1}. Ignore the curvature of the ocean surface. §

Generalization
This formulation leads to a natural generalization to N-dimensional vectors. See Appendix A.

2.2 Basis vectors

Introduce three *basis vectors* $\hat{i}$, $\hat{j}$ and $\hat{k}$ along the x, y and z directions, each with unit length; the hat symbol denotes a *unit vector.* The representations of these basis vectors in the manner of (1) would be

$$\begin{aligned} \hat{i} &= (1,0,0) \\ \hat{j} &= (0,1,0) \\ \hat{k} &= (0,0,1) \end{aligned} \tag{3}$$

2.3 Expression in terms of basis vectors

It then follows trivially that

$$\boxed{\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}} \tag{4}$$

Vector addition and scalar multiplication follow the "obvious" rules of arithmetic, e.g.,

$$\begin{aligned} \vec{c} &= \vec{a} + \vec{b} \\ &= \left(a_x\hat{i} + a_y\hat{j} + a_z\hat{k}\right) + \left(b_x\hat{i} + b_y\hat{j} + b_z\hat{k}\right) \\ &= (a_x{+}b_x)\,\hat{i} + (a_y{+}b_y)\,\hat{j} + (a_z{+}b_z)\,\hat{k} \end{aligned} \tag{5}$$

from which (2) follows immediately.

2.4 Improved notation

Systematic labelling
Call the x, y, z axes the 1-axis, 2-axis, 3-axis respectively, and relabel the components as

$$a_x, a_y, a_z \quad \mapsto \quad a_1, a_2, a_3$$

Moreover, call the basis vectors $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$; in other words

$$\hat{i}, \hat{j}, \hat{k} \quad \mapsto \quad \hat{e}_1, \hat{e}_2, \hat{e}_3$$

Then (4) can be written more neatly as

$$\begin{aligned} \vec{a} &= a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3 \\ &= \sum_{i=1}^{3} a_i\hat{e}_i \end{aligned} \tag{6}$$

Any number of dimensions
Obviously, the corresponding formula in N dimensions is

$$\boxed{\vec{a} = \sum_{i=1}^{N} a_i\hat{e}_i} \tag{7}$$

Summation convention
Such sums occur so often that the following convention is often adopted: If the same index occurs twice in the same term, it is understood to be a dummy index to be summed over. Thus

$$a_i\hat{e}_i \quad \text{means} \quad \sum_{i=1}^{N} a_i\hat{e}_i$$

Therefore

$$\boxed{\vec{a} = a_i\hat{e}_i}$$

In those rare cases when we do not want to sum, that should be indicated explicitly. But in the rest of this module, for the benefit of beginning students, the summation signs will be kept explicitly.

3 Dot product

3.1 Physical motivation

The following concept is familiar: If a force $\vec{F}$ moves an object through a displacement $\vec{s}$, the work done is (**Figure 7**)

$$W = F_{\parallel}\, s = F\, s\, \cos\gamma \tag{8}$$

where $F_{\parallel}$ is the component of $\vec{F}$ parallel to the displacement, and γ is the angle between $\vec{F}$ and $\vec{s}$; the component of $\vec{F}$ perpendicular to the displacement does no work.

One is then motivated to define the *dot product*, also called the *scalar product*, between two vectors $\vec{a}$ and $\vec{b}$ as

$$\boxed{\vec{a}\cdot\vec{b} = a\, b\, \cos\gamma} \tag{9}$$

where a, b are the respective magnitudes, and γ is the angle between the two vectors (**Figure 8**).

Similar to (8) the dot product can also be written as

$$\vec{a}\cdot\vec{b} = a_{\parallel}\, b \tag{10}$$

where $a_{\parallel}$ is the component of $\vec{a}$ along $\vec{b}$. We can also project $\vec{b}$ along $\vec{a}$.

3.2 Main properties

Invariance
The definition (9) makes no reference to the coordinate axes. So the dot product remains the same when the axes are rotated.

Symmetry
The definition (9) is symmetric under the interchange of the two vectors:

$$\boxed{\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}} \tag{11}$$

Linearity
Linearity means two properties. First, if s is a scalar, then

$$(s\,\vec{a})\cdot\vec{b} = s\,(\vec{a}\cdot\vec{b}) \tag{12}$$

which is obvious from (9) because the magnitude of $s\,\vec{a}$ is of course $s\,a$.

Second, given vectors $\vec{a}_1$ and $\vec{a}_2$,

$$(\vec{a}_1 + \vec{a}_2)\cdot\vec{b} = \vec{a}_1\cdot\vec{b} + \vec{a}_2\cdot\vec{b} \tag{13}$$

To prove this statement refer to **Figure 9**, in which $\vec{a} = \vec{a}_1 + \vec{a}_2$. From (10), each dot product can be expressed in terms of a projection along $\vec{b}$: in obvious notation

$$\begin{aligned} \vec{a}_1\cdot\vec{b} &= a_{1\parallel}\, b \\ \vec{a}_2\cdot\vec{b} &= a_{2\parallel}\, b \\ \vec{a}\cdot\vec{b} &= a_{\parallel}\, b \end{aligned}$$

while one sees from the diagram that the projections add:

$$a_{1\parallel} + a_{2\parallel} = a_{\parallel}$$

which then proves (13).

Putting (12) and (13) together, we have

$$\boxed{(s_1\,\vec{a}_1 + s_2\,\vec{a}_2)\cdot\vec{b} = s_1\,(\vec{a}_1\cdot\vec{b}) + s_2\,(\vec{a}_2\cdot\vec{b})} \tag{14}$$

In (13) and (14), the + sign on the LHS denotes vector addition; the one on the RHS denotes scalar addition.

The property of linearity or additivity can likewise be written for the second factor.

Dot product among basis vectors
Since the basis vectors are orthogonal unit vectors,

$$\begin{aligned} \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} &= 1 \\ \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} &= 0 \end{aligned}$$

or, using the the notation introduced in Section 2.4:

$$\boxed{\hat{e}_i\cdot\hat{e}_j = \delta_{ij}} \tag{15}$$

3.3 Expression in terms of components

Next, we want to express $\vec{a}\cdot\vec{b}$ in terms of the components of the two vectors. Here the relevant formula is derived in a special case, and the more general proof is given in Appendix A.

Figure 10 shows vectors $\vec{a}$ and $\vec{b}$ in the x–y plane, making angles α and β with respect to the x axis; the angle between the two vectors is (up to an irrelevant sign) $\gamma = \beta - \alpha$. From (9),

$$\begin{aligned} \vec{a}\cdot\vec{b} &= ab\,\cos(\beta-\alpha) \\ &= ab\,(\cos\beta\cos\alpha+\sin\beta\sin\alpha) \\ &= (a\cos\alpha)(b\cos\beta)+(a\sin\alpha)(b\sin\beta) \\ &= a_x b_x + a_y b_y \end{aligned} \tag{16}$$

It is easy to guess the corresponding formula in 3D:

$$\vec{a}\cdot\vec{b} \quad = \quad a_x b_x + a_y b_y + a_z b_z$$

or in the compact notation introduced

$$\boxed{\vec{a}\cdot\vec{b} = \sum_i a_i b_i} \tag{17}$$

3.4 Algebraic formulation

We now reformulate the dot product in algebraic terms. Define the product $\vec{a}\cdot\vec{b}$ to be symmetric (see (11)) and linear (see (14)), and moreover with the dot product between basis vectors given by (15). Then, for any two vectors

$$\begin{aligned} &\vec{a}\cdot\vec{b} \\ =\ &\left(\sum_i a_i\,\hat{e}_i\right)\cdot\left(\sum_j b_j\,\hat{e}_j\right) \\ =\ &\sum_{ij} a_i b_j\ (\hat{e}_i\cdot\hat{e}_j) \\ =\ &\sum_{ij} a_i b_j\,\delta_{ij} = \sum_i a_i b_i \end{aligned}$$

proving (17). The above has been deliberately written to be valid in any number of dimensions.

Problem 3
Find the angle γ between the following vectors. $\vec{a} = 3\hat{\imath} + 4\hat{\jmath}$ and $\vec{b} = 5\hat{\jmath} + 6\hat{k}$. §

Problem 4
The radius of the earth is 6370 km, and the positions of Hong Kong (H) and Buenos Aires (B) are as follows:[4] $H = 22.4\,^\circ$N, $114.1\,^\circ$E; $B = 34.6\,^\circ$S, $58.4\,^\circ$W. (a) Find the angle between OH and OB, where O is the center of the earth. (b) Find the shortest flying distance from H to B, assuming the flight altitude is negligible. §

[4] See a corresponding Problem in the module on Coordinates.

3.5 Logic of the definition

The heuristic approach adopted so far suggests the following chain of logic. (a) The perpendicular component of force does no work. (b) Therefore work is defined in terms of the parallel component, and the dot product involves projecting one vector onto the other. (c) The dot product so defined is linear (i.e., additive).

But this logic is *wrong* — even though it might be pedagogically suggestive. How do we *know* that the perpendicular component of force does no work?

The actual logic is the other way round. (a) We want a definition of dot product and of work that is additive. Otherwise the sum of the work done by several forces (adding up several scalars) would not be equal to the work done by the net force (adding up several vectors). (b) The dot product is defined to be additive. Therefore we define work using this dot product. (c) As a result of this *definition*, the perpendicular component of force does no work.

To emphasize this point, suppose we define work to be the product of the magnitudes: $W' = Fs$. Consider two forces $\vec{F}_1 = \hat{\imath}$, $\vec{F}_2 = -\hat{\imath}$ and a displacement $\vec{s} = \hat{\jmath}$. The individual values are $W_1' = 1$, $W_2' = 1$. But the total force $\vec{F} = 0$ gives $W' = 0 \neq W_1' + W_2'$. You can of course define work this way, but the resultant concept would not be of much use.

4 Cross product

4.1 Physical motivation

Turning moment
It is well known that the *turning moment* or *torque* due to a force $\vec{F}$ is given by

$$\text{torque} \quad = \quad \text{moment arm} \times \text{force}$$

as shown in **Figure 11a**. More generally, if the force and the moment arm are not perpendicular, then only the component of force perpendicular to the moment arm contributes; see **Figure 11b**.

So, in general, the torque has a magnitude

$$\tau \quad = \quad r\,F_\perp = rF\sin\gamma \tag{18}$$

where r is the moment arm, $F_\perp$ is the component of $\vec{F}$ perpendicular to the moment arm, and γ is the angle between the moment arm and the force.

Since we are talking about the magnitude, in the above we should take γ to be positive.

Torque as a vector
It is convenient to regard the torque as a vector. In **Figures 11a**, **11b** imagine the moment arm $\vec{r}$ and the force $\vec{F}$ to be in the x–y plane. The resultant torque tends to rotate the object about the z-axis, and that is the direction assigned to $\vec{\tau}$ as a vector.

The *sense* of $\vec{\tau}$ is defined by the *right-hand rule* convention: If the thumb of the right hand points along $\vec{\tau}$, then the body tends to rotate in the direction described by the other fingers.

Definition in terms of cross product
Therefore the *cross product* $\vec{\tau} = \vec{r} \times \vec{F}$ is defined to have these properties. In general, given two vectors $\vec{a}$ and $\vec{b}$,

$$\vec{c} \;=\; \vec{a} \times \vec{b}$$

is defined as follows.

- Its direction is given by the right-hand rule: When the fingers of the right hand curl from the first vector $\vec{a}$ to the second vector $\vec{b}$, the thumb gives the direction of $\vec{c}$.
- The magnitude is given by

  $$\boxed{c = ab \sin \gamma} \tag{19}$$

  where γ is the angle measured from the first vector to the second.

4.2 Main properties

Invariance
The definition above makes no reference to the coordinate axes. In any coordinate system, the result is the same vector. So the cross product remains the same when the axes are rotated.

Antisymmetry
In applying the right-hand rule, the fingers go from the first vector to the second vector. So if the two vectors are interchanged, the direction of the cross product is reversed:

$$\boxed{\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}} \tag{20}$$

Linearity
Linearity means two properties. First, if s is a scalar, then obviously

$$(s\,\vec{a}) \times \vec{b} \;=\; s\,(\vec{a} \times \vec{b}) \tag{21}$$

Second, given vectors $\vec{a}_1$ and $\vec{a}_2$,

$$(\vec{a}_1 + \vec{a}_2) \times \vec{b} \;=\; \vec{a}_1 \times \vec{b} + \vec{a}_2 \times \vec{b} \tag{22}$$

To prove this statement, at least in the special case where all the three vectors lie in one plane, refer to **Figure 12**, in which $\vec{a} = \vec{a}_1 + \vec{a}_2$. The cross products are all perpendicular to the page, with magnitudes given in obvious notation as

$$\begin{aligned} |\vec{a}_1 \times \vec{b}| &= a_{1\perp}\, b \\ |\vec{a}_2 \times \vec{b}| &= a_{2\perp}\, b \\ |\vec{a} \times \vec{b}| &= a_{\perp}\, b \end{aligned}$$

while from the diagram it is obvious that the projections add:

$$a_{1\perp} + a_{2\perp} \;=\; a_{\perp}$$

which then proves (22).

Combining (21) and (22) gives

$$\boxed{(s_1\,\vec{a}_1 + s_2\,\vec{a}_2) \times \vec{b} = s_1\,(\vec{a}_1 \times \vec{b}) + s_2\,(\vec{a}_2 \times \vec{b})} \tag{23}$$

The property of linearity also applies to the second factor.

Cross product among basis vectors
The cross products among the basis vectors are given by

$$\begin{aligned} &\hat{\imath} \times \hat{\imath} = 0 \quad , \quad \hat{\jmath} \times \hat{\jmath} = 0 \quad , \quad \hat{k} \times \hat{k} = 0 \\ &\hat{\imath} \times \hat{\jmath} = \hat{k} \quad , \quad \hat{\jmath} \times \hat{k} = \hat{\imath} \quad , \quad \hat{k} \times \hat{\imath} = \hat{\jmath} \end{aligned} \tag{24}$$

Right-handed coordinate system
In writing $\hat{\imath} \times \hat{\jmath} = \hat{k}$ and not $-\hat{k}$, etc., we are assuming that the coordinate system is *right-handed*: If the x and y axes are as shown (**Figure 13**), then the z axis must point out of the page and not into the page. A common choice is the x axis points east, the y axis points north, and the z axis points up.

Levi-Civita symbol
The Levi-Civita symbol ϵ_{ijk} is defined as follows:

$$\begin{aligned} \epsilon_{123} = \epsilon_{231} = \epsilon_{312} &= 1 \\ \epsilon_{321} = \epsilon_{213} = \epsilon_{132} &= -1 \end{aligned}$$

while $\epsilon_{ijk} = 0$ if any two indices are equal. Then denoting the basis vectors as $\hat{e}_i$ as usual, we see that (24) can be summarized as

$$\boxed{\hat{e}_i \times \hat{e}_j = \sum_k \epsilon_{ijk}\hat{e}_k} \tag{25}$$

It is easy to see that for a 3×3 matrix A with elements A_{ij},

$$\det A = \sum_{ijk} \epsilon_{ijk}\, A_{1i}A_{2j}A_{3k} \tag{26}$$

Problem 5
Verify that ϵ_{ijk} is cyclic: $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}$ and also verify the formula (26). §

4.3 Expression in terms of components

Next, express $\vec{a} \times \vec{b}$ in terms of the components. Here the relevant formula is derived only in a special case, and the more general proof is given in Section 4.4.

Figure 14 shows vectors $\vec{a}$ and $\vec{b}$ in the x–y plane, making angles α and β with respect to the x-axis; the angle between the two vectors is $\gamma = \beta - \alpha$. From the figure, $\vec{c} = \vec{a} \times \vec{b}$ points in the z direction, so c_z in this case is equal to the magnitude, and given by

$$\begin{aligned} c_z &= |c| = ab\sin\gamma = ab\sin(\beta - \alpha) \\ &= ab\,(\sin\beta\cos\alpha - \cos\beta\sin\alpha) \\ &= (a\cos\alpha)(b\sin\beta) - (a\sin\alpha)(b\cos\beta) \\ &= a_x b_y - a_y b_x \end{aligned}$$

The general case is readily guessed:

$$\begin{aligned} c_x &= a_y b_z - a_z b_y \\ c_y &= a_z b_x - a_x b_y \\ c_z &= a_x b_y - a_y b_x \end{aligned}$$

which can be written compactly as

$$\boxed{c_i = \sum_{jk} \epsilon_{ijk}\, a_j b_k} \tag{27}$$

The result can be remembered easily as:

$$\vec{a} \times \vec{b} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{28}$$

Problem 6
Using the above formula, verify that $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$. §

Problem 7
Using (28), evaluate $\vec{a} \times \vec{b}$ if $\vec{a} = 3\hat{\imath} + 4\hat{\jmath}$ and $\vec{b} = 5\hat{\jmath} + 6\hat{k}$. §

Problem 8
Consider a parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$. Show that its area is given by $|\vec{a} \times \vec{b}|$. §

4.4 Algebraic formulation

The cross product can be reformulated in algebraic terms. Define the product $\vec{a} \times \vec{b}$ to be antisymmetric (see (20)) and linear (see (23)), and moreover with the cross product between basis vectors given by (25). Then,

$$\begin{aligned} \vec{c} &= \vec{a} \times \vec{b} \\ &= \left(\sum_j a_j\,\hat{e}_j\right) \times \left(\sum_k b_k\,\hat{e}_k\right) \\ &= \sum_{jk} a_j b_k\, (\hat{e}_j \times \hat{e}_k) \\ &= \sum_{jk} a_j b_k \left(\sum_i \epsilon_{jki}\,\hat{e}_i\right) \\ &= \sum_i \left(\sum_{jk} \epsilon_{jki}\, a_j b_k\right) \hat{e}_i \end{aligned} \tag{29}$$

The bracket multiplying $\hat{e}_i$ should be identified as c_i, which then gives (27) upon using the cyclic property of ϵ_{ijk}.

4.5 Only in 3D

The essence of the cross product is contained in the formula $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$, i.e., the answer is the "remaining" basis vector. It is obvious that this idea cannot be generalized to more than 3 dimensions. Likewise, the Levi-Civita symbol is only defined for 3D. (However, in 4D there is the analogous object with 4 indices $\epsilon_{ilk\ell}$, and likewise in higher dimensions.)

4.6 Use in physical laws

Many physical laws are formulated using either the right hand or the left hand, for example, in magnetism:

- The magnetic force on a current-carrying wire (in the old days often expressed by the Fleming left hand rule).
- The EMF around a loop caused by a changing magnetic flux (Faraday's law).
- The magnetic field caused by a long wire (Ampere's law) or by a short wire segment (Biot–Savart law).

In advanced studies of physics, all such laws are expressed using the cross product.

4.7 Parity

The laws of physics often involve the cross product, and through it, the *right*-hand rule. Can we use the left-hand rule instead?

Example in electromagnetism
Consider the following situation.

- A wire produces a magnetic field $\vec{B}$:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{\ell} \times \vec{r}}{r^3}$$

  where $d\vec{\ell}$ is an element of the wire carrying current I, and $\vec{r}$ is the displacement from the wire element to the observation point.[5]

- A charged particle moving at velocity $\vec{v}$ then experiences a force

$$\vec{F} = q\,\vec{v} \times \vec{B}$$

The right-hand rule is used once in each step. If we were to use the left hand instead, the final answer for $\vec{F}$ and for the resultant acceleration $\vec{a} = \vec{F}/m$ (which is observable) is unchanged. It is true that $\vec{B}$ changes sign, but $\vec{B}$ is not directly observable, so that is alright.

In fact, in all of electromagnetism, we can use the left hand instead. The same is true of the strong interaction and gravity.

[5] Students do not have to know the details, except that the cross product appears once.

Mirror world
Viewed in a mirror, the right hand becomes the left hand. So an equivalent statement is: For the electromagnetic, strong and gravitational interactions, the real world and the mirror world satisfy the same laws of physics. Physicists say that in these cases *parity is conserved.*

Non-conservation of parity in weak interactions
The above statement seems so "obvious" that people took it for granted as universally true. In trying to resolve some puzzling experimental results, TD Lee and CN Yang proposed in 1956 that perhaps such is not the case for the weak interactions (e.g., β decays).

A sample of ^{60}Co at very low temperatures is placed at the center of a circular current coil in the horizontal plane. The "up" and "down" directions are specified relative to the current loop by the right-hand rule. The electrons emitted in the β decay were observed, in a delicate experiment by CS Wu et al. They found an up/down asymmetry of electrons, proving that parity is not conserved. Lee and Yang were awarded the 1957 Nobel Prize in physics.

5 Some identities

Start with the following identity

$$\boxed{\sum_i \epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}} \qquad (30)$$

This identity can be proved by simply checking the finite number of possibilities. For example take the case $j = 2$, $k = 3$, $m = 2$, $n = 3$.

$$\begin{aligned} \text{LHS} &= \sum_i \epsilon_{i23}\epsilon_{i23} = \epsilon_{123}\epsilon_{123} = 1 \\ \text{RHS} &= \delta_{22}\delta_{33} - \delta_{23}\delta_{32} = 1 - 0 \end{aligned}$$

Another identity is

$$\boxed{\sum_{ij} \epsilon_{ijk}\epsilon_{ijn} = 2\delta_{kn}} \qquad (31)$$

Again this identity can be proved by simply checking the finite number of possibilities. Or, starting from (30), set $m = j$ and sum over it.

Consider the vector

$$\vec{v} \quad = \quad (\vec{a} \times \vec{b}) \times \vec{c}$$

and denote $\vec{d} = \vec{a} \times \vec{b}$. Then

$$\begin{aligned}
v_i &= \sum_{jk} \epsilon_{ijk} d_j c_k \\
&= \sum_{jk} \epsilon_{ijk} \left(\sum_{mn} \epsilon_{jmn} a_m b_n \right) c_k \\
&= \sum_{jkmn} \epsilon_{ijk} \epsilon_{jmn} a_m b_n c_k \\
&= \sum_{jkmn} (\delta_{km}\delta_{in} - \delta_{kn}\delta_{im})\, a_m b_n c_k \\
&= b_i \left(\sum_k a_k c_k \right) - a_i \left(\sum_k b_k c_k \right) \\
v_i &= (\vec{a} \cdot \vec{c})\, b_i - (\vec{b} \cdot \vec{c})\, a_i \\
\vec{v} &= (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{b} \cdot \vec{c})\, \vec{a}
\end{aligned}$$

giving the identity

$$\boxed{(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{b} \cdot \vec{c})\, \vec{a}} \qquad (32)$$

Finally, with proof left as an exercise

$$\boxed{(\vec{a} \times \vec{b}) \cdot \vec{c} = \sum_{ijk} \epsilon_{ijk} a_i b_j c_k} \qquad (33)$$

This identity also implies that the LHS of (33) remains unchanged under cyclic permutation of the three vectors.

Problem 9
A parallelepiped has adjacent sides $\vec{a}$, $\vec{b}$, $\vec{c}$. Show that its volume is given by (33) up to a sign. This provides another proof that cyclic permutation of the three vectors does not change this triple product. §

6 Vectors under rotation

6.1 Rotation matrix

Let $\vec{a} = (a_x, a_y, a_z)$ be a displacement vector, say the coordinates of a point A measured from the origin. If now the axes are rotated, the components will change to $\vec{a}' = (a'_x, a'_y, a'_z)$.[6] The new coordinates are related *linearly* to the old ones, so the most general relation is

$$a'_i \quad = \quad \sum_j R_{ij}\, a_j \qquad (34)$$

This relation can be expressed compactly if $\vec{a}$ is regarded as a column vector

$$[a] \quad = \quad \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

and R_{ij} as the elements of a matrix

$$[R] \quad = \quad \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}$$

Then (34) is just matrix multiplication

$$[a'] \quad = \quad [R]\,[a] \qquad (35)$$

The matrix $[R]$ is called the *rotation matrix*.

Example
Let $\vec{a}$ be a displacement vector in the x–y plane, making an angle ϕ with the x-axis; thus

$$a_x = a\,\cos\phi \quad , \quad a_y = a\,\sin\phi$$

Now rotate the axes backwards by an angle α; or equivalently rotate the vector forwards by an angle α. The angle between the vector and the new x' axis is

$$\phi' \quad = \quad \phi + \alpha$$

Therefore the new components are

$$\begin{aligned}
a'_x &= a\,\cos(\phi{+}\alpha) \\
&= a\,(\cos\phi\,\cos\alpha - \sin\phi\,\sin\alpha) \\
&= \cos\alpha\,(a\cos\phi) - \sin\alpha\,(a\sin\phi) \\
&= \cos\alpha\; a_x - \sin\alpha\; a_y \\
a'_y &= a\,\sin(\phi{+}\alpha) \\
&= a\,(\sin\phi\,\cos\alpha + \cos\phi\,\sin\alpha) \\
&= \sin\alpha\,(a\cos\phi) + \cos\alpha\,(a\sin\phi) \\
&= \sin\alpha\; a_x + \cos\alpha\; a_y
\end{aligned}$$

[6] Strictly speaking, it is the same vector referred to new axes x' etc. So purists would write $a_{x'}$ etc., but that is usually regarded as too cumbersome.

which is an example of (35) with

$$[R] \quad = \quad \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (36)$$

in which the trivial transformation $a'_z = a_z$ has been incorporated.

It is important to note (as is evident from this example) that $[R]$ depends only on the transformation (i.e., how the axes are rotated) and does not depend on the particular vector $\vec{a}$ that is under consideration.

6.2 Formal definition of a vector

Any three quantities $[a]$ that transforms in the same way as (35) is a vector. On the other hand, any one quantity that remains unchanged under rotation of axes is a scalar. This definition does not rely on any notion of "direction". Also, note that the quantities constructed in the manner of Section 1.4 are guaranteed to be vectors in this sense.

6.3 Condition on rotation matrix

Not every 3×3 matrix $[R]$ qualifies as a rotation matrix. It must preserve dot products. Thus

$$\begin{aligned} a'_i &= \sum_m R_{im}\, a_m \\ b'_i &= \sum_n R_{in}\, b_n \\ \vec{a}' \cdot \vec{b}' &= \sum_i a'_i b'_i \\ &= \sum_{mn} \left(\sum_i R_{im} R_{in} \right) a_m b_n \end{aligned}$$

But this must equal, as an identity,

$$\vec{a} \cdot \vec{b} \quad = \quad \sum_{mn} (\delta_{mn})\; a_m b_n$$

Hence the matrix $[R]$ must satisfy

$$\boxed{\sum_i R_{im} R_{in} = \delta_{mn}} \qquad (37)$$

We can write, in terms of the transposed matrix $[R^{\mathrm{T}}]$

$$R_{im} \quad = \quad R^{\mathrm{T}}_{mi}$$

so (37) becomes

$$\sum_i R^{\mathrm{T}}_{mi} R_{in} = \delta_{mn}$$

Multiplication and summation over neighboring indices is just matrix multiplication, so (37) can be written as the matrix equation

$$\boxed{[R^{\mathrm{T}}]\,[R] = [I]} \qquad (38)$$

where $[I]$ is the identity matrix. A matrix $[R]$ that satisfies this condition is said to be *orthogonal.*

In this subsection, we have deliberately omitted the upper limit of the summation over dummy indices, and the entire formalism is valid in any number of dimensions.

Problem 10
Check explicitly that the rotation matrix in (36) satisfies (38). §

Appendix

A Vectors in N-dimensional space

Standard formulation
These appendices generalize the concept of vectors and vector spaces. Here, we first generalize vectors to N dimensions.

Consider the set of all ordered N-tuples of real numbers:

$$\vec{a} \quad = \quad (a_1, a_2, \ldots, a_N) \qquad (39)$$

Define addition and multiplication by a real number (a scalar) in the obvious way, component by component.

The length or *norm* $|\vec{a}|$ is defined by[7]

$$|\vec{a}|^2 \quad = \quad \sum_i |a_i|^2 \qquad (40)$$

which has the property that it is non-negative, and zero only it $\vec{a}$ is identically zero. This norm is obviously unchanged under a rotation of axes.

[7] The absolute value sign on the RHS is included only to anticipate generalization to complex vectors.

The dot product between two vectors is defined by the equation

$$|\vec{a}+\vec{b}|^2 \quad = \quad |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 \tag{41}$$

Since the three terms with norms are unchanged under rotation, so the dot product is also unchanged under rotation.

By putting (40) into (41), it is easily shown that

$$\vec{a}\cdot\vec{b} \quad = \quad \sum_i a_i b_i$$

from which its linearity property follows trivially.

Problem 11
Show that the dot product is bounded by

$$-ab \le \vec{a}\cdot\vec{b} \le ab$$

or in other words $|\vec{a}\cdot\vec{b}|/ab \le 1$. Hint: Consider

$$f(t) \quad = \quad |\vec{a}+t\,\vec{b}|^2$$

Since $f(t) \ge 0$ for all t, there is a condition on the discriminant of the quadratic. §

The angle γ between two vectors a and b is defined by

$$\cos\gamma \quad = \quad \frac{\vec{a}\cdot\vec{b}}{ab}$$

By Problem 11, the RHS is between -1 and $+1$, and can be identified as a cosine. How do we know that this definition agrees with the usual notion of the angle? Since dot products are invariant under rotation of axes, we can perform a rotation to put $\vec{a}$ and $\vec{b}$ onto the x–y plane. Then go back to the derivation of (16), which establishes the agreement with the usual notion of angle.

Example 1
There are N students in a class, and they take an examination with questions (a), (b), (c), ... Let a_i, b_i be the scores of student i on question (a), (b) etc. Then all the scores can be encoded as the vectors[8]

$$\begin{aligned}\vec{a} \quad &= \quad (a_1, a_2, \ldots, a_N)\\ \vec{b} \quad &= \quad (b_1, b_2, \ldots, b_N)\end{aligned}$$

etc. These are vectors in N-dimensional space. The teacher decides to compute the total score (t) by adding the questions with weights w_a, w_b, etc. This is then expressed as a vector addition

$$\vec{t} \quad = \quad w_a\vec{a} + w_b\vec{b} + \ldots$$

One often defined a *correlation* between two questions as, e.g.,

$$C_{ab} \quad = \quad \frac{\vec{a}\cdot\vec{b}}{ab}$$

which is essentially $\cos\gamma$. By the results just proved, this must lie in the range -1 to $+1$. If the value is $+1$, it means the two vectors are perfectly aligned; the scores on question (a) and the scores on question (b) are perfectly correlated — the one predicts the other. §

Example 2
Consider the set of 3×3 matrices M, with addition and scalar multiplication in the usual way. Define the norm as[9]

$$\|M\|^2 \quad = \quad \sum_{ij} M_{ij}^2 = \mathrm{tr}\;[M^{\mathrm{T}}]\,[M]$$

where tr denotes the trace of a matrix. This is a 9-dimensional vector space. It is easy to generalize to larger matrices. §

More general norm
Even more generally, we can define the norm as follows:

$$\begin{aligned}|\vec{a}|^2 \quad &= \quad \sum_{ij} g_{ij}\,a_i\,a_j\\ &= \quad [a^{\mathrm{T}}]\,[g]\,[a]\end{aligned} \tag{42}$$

where $[g]$ is a fixed $N\times N$ symmetric matrix with the property that (42) is non-negative, and zero only if $\vec{a}$ is identically zero.[10]

Everything in this Appendix is still valid. The only amendment is that a rotation is now defined as a linear transformation such as (35) which preserves (42).

Problem 12
Find the condition on $[R]$ in this case. §

Complex vector space
A further generalization is to go to *complex* vectors,

[8] In fact, it is usual to remove the average value.

[9] In this case it is conventional to use the double vertical bar.

[10] If you have studied linear algebra, this means that all the eigenvalues of $[g]$ have to be positive.

i.e., N-tuples such as (39) where the components are allowed to be complex numbers. Now we also allow multiplication by complex scalars s. The dot product is now defined as

$$\vec{a} \cdot \vec{b} \quad = \quad \sum_i a_i^* b_i$$

where * denotes complex conjugate. This is linear in the second variable, but *conjugate linear* in the first variable:

$$\begin{aligned} \vec{a} \cdot (s\,\vec{b}) &= s\,(\vec{a} \cdot \vec{b}) \\ (s\,\vec{a}) \cdot \vec{b} &= s^*\,(\vec{a} \cdot \vec{b}) \end{aligned}$$

The ratio $\vec{a} \cdot \vec{b}/(ab)$ is now a complex number whose magnitude is at most unity. Of course it can no longer be interpreted as some $\cos\gamma$ for a real γ.

Apart from these modifications, everything goes through as before. Proofs are left as exercises.

B Abstract linear space

We can go one step further. Let ψ be any set of objects which can be added and multiplied by scalars, and for which a positive-definite norm can be defined. As an example, let ψ be any function[11] of a continuous variable x on the interval say $a \le x \le b$, say vanishing at the endpoints. They can be added, and can be multiplied by a scalar in the obvious way. The norm is defined as

$$\|\psi\|^2 \quad = \quad \int_a^b |\psi(x)|^2\,dx \tag{43}$$

This defines a *linear space*, with the dot product between two functions:

$$\langle\phi|\psi\rangle \quad = \quad \int_a^b \phi(x)^* \psi(x)\,dx \tag{44}$$

where we have introduced the Dirac notation for writing the dot product — in this case more commonly called an inner product. Since the LHS of (44) is a bracket,

- $\langle\phi|$ is called a bra vector.
- $|\psi\rangle$ is called a ket vector.

[11]Reasonably smooth, at least in the sense that (43) can be defined.

C Wedge product

Motivation

The cross product $\vec{a} \times \vec{b}$ has the following properties.

- The *magnitude* can be interpreted as the area of the parallelogram with sides $\vec{a}$, $\vec{b}$. This is a concept in the 2D plane of the two vectors, and makes sense no matter how many other dimensions there are.
- The *direction* is associated with the normal to that plane. In 3D, the normal is unique (up to a sign, which is fixed by the convention of right-hand rule). But in N dimensions with $N > 3$, the normal to a plane is not unique — there are $N-2$ independent choices. For example, in 4D, if the plane is defined by $\hat{e}_1$ and $\hat{e}_2$, then any linear combination of $\hat{e}_3$ and $\hat{e}_4$ is normal to the plane. Therefore the cross product cannot be defined for space of N dimensions when $N \neq 3$.

But suppose we think about such a product not as a vector (i.e., another 1D object) but simply as an area (i.e., a 2D object) in the plane of the two vectors. Everything is referenced to that plane, irrespective of how many other dimensions there are.

With this in mind, we define a *wedge product* between two vectors, denoted as $\vec{a} \wedge \vec{b}$, to denote such as area. What properties should it have?

Linearity

Multiplication by a scalar is obvious and we only consider the equality

$$(\vec{a}_1 + \vec{a}_2) \wedge \vec{b} \quad = \quad \vec{a}_1 \wedge \vec{b} + \vec{a}_2 \wedge \vec{b} \tag{45}$$

which would be a nice property to have. Let us check the consistency with the interpretation in terms of area. The two terms on the RHS are illustrated by the two shaded areas in **Figure 15a**, while the LHS is given by the shaded area in **Figure 15b**, in which $\vec{a} = \vec{a}_1 + \vec{a}_2$. The equality of area is obvious geometrically.

Antisymmetry

The area shown in **Figure 16a** is $\hat{e}_1 \wedge \hat{e}_2$. By a continuous rotation in the plane, it must be equal to the area in **Figure 16b**, which is $(-\hat{e}_2) \wedge \hat{e}_1$. This shows that the wedge product should satisfy

$$\vec{a} \wedge \vec{b} \quad = \quad -\vec{b} \wedge \vec{a} \tag{46}$$

Thus these wedge products represent *signed* areas.

A corollary is

$$\vec{a} \wedge \vec{a} \quad = \quad 0 \qquad (47)$$

which is consistent with the interpretation in terms of area.

Representation in terms of a basis

Let

$$\begin{aligned} \vec{a} &= a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3 \\ \vec{b} &= b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3 \end{aligned}$$

Using linearity, $\vec{a} \wedge \vec{b}$ can be expressed as the sum of 9 terms such as $a_1 b_2 \, \hat{e}_1 \wedge \hat{e}_2$. The three diagonal terms vanish by (47) while the other six terms can be combined in pairs using (46), giving finally

$$\begin{aligned} & \vec{a} \wedge \vec{b} \\ = \; & (a_1 b_2 - a_2 b_1) \, \hat{e}_1 \wedge \hat{e}_2 \\ & + (a_1 b_3 - a_3 b_1) \, \hat{e}_1 \wedge \hat{e}_3 \\ & + (a_2 b_3 - a_3 b_2) \, \hat{e}_2 \wedge \hat{e}_3 \end{aligned} \qquad (48)$$

The geometric interpretation is as follows. The LHS represents a parallelogram in 3D. The RHS represents the projections of this parallelogram onto the 1−2, 1−3 and 2−3 planes, each projection itself being a parallelogram.

It is important to stress that the above is still valid if there is a fourth dimension, so long as $\vec{a}$ and $\vec{b}$ lie within the first 3 dimensions.

Problem 13

Consider the special case where $\vec{a}$ and $\vec{b}$ lie in the 1−2 plane, i.e., $a_3 = b_3 = 0$. Show that

$$\vec{a} \wedge \vec{b} \quad = \quad (ab \sin\gamma) \, \hat{e}_1 \wedge \hat{e}_2$$

where γ is the angle between the two vectors. This formula means: The parallelogram formed by $\vec{a}$ and $\vec{b}$ is A times the unit square formed by $\hat{e}_1$ and $\hat{e}_2$, where $A = ab \sin\gamma$. §

Problem 14

Write out the analogous representation if the two vectors also have the 4th dimension, i.e., including terms $a_4 \hat{e}_4$ and $b_4 \hat{e}_4$. §

In N dimensions, any $\vec{a} \wedge \vec{b}$ can be represented as

$$\vec{a} \wedge \vec{b} \quad = \sum_{1 \le i < j \le N} (a_i b_j - a_j b_i) \, \hat{e}_i \wedge \hat{e}_j \qquad (49)$$

where there are $N(N-1)/2$ terms.

Higher-order wedge products

In 3D space, three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ define a parallelepiped, and it is convenient to define the triple wedge product

$$\vec{a} \wedge \vec{b} \wedge \vec{c}$$

to represent the signed volume. The triple wedge product is linear, and antisymmetric upon interchange of any two vectors. By using these properties, it is simple to show that

$$\begin{aligned} \vec{a} \wedge \vec{b} \wedge \vec{c} &= V \, \hat{e}_1 \wedge \hat{e}_2 \wedge \hat{e}_3 \\ V &= \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \end{aligned} \qquad (50)$$

- If the 3 vectors lie in 3D, the answer must be expressible in terms of only one elementary unit volume $\hat{e}_1 \wedge \hat{e}_2 \wedge \hat{e}_3$.
- However, the above still holds if there are other dimensions.
- The same idea can be generalized to the hypervolume associated with 4 vectors in 4D space, etc.

Building up lines, surfaces and volumes

A curve can be thought of as being made up of many short linear segments, each represented as a vector $\vec{a}$ (**Figure 17a**). A surface can be thought of as being made up of many small parallelograms, each represented as $\vec{a} \wedge \vec{b}$ (**Figure 17b**). A volume can be thought of as being made up of many small parallelepipeds $\vec{a} \wedge \vec{b} \wedge \vec{c}$ (**Figure 17c**). In all these cases, the vectors should be regarded as infinitesimal.

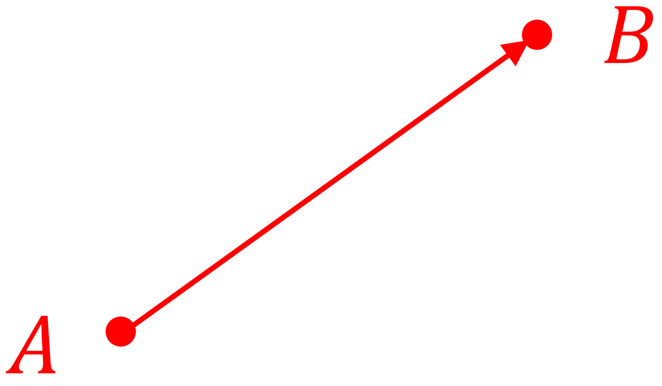

Figure 1

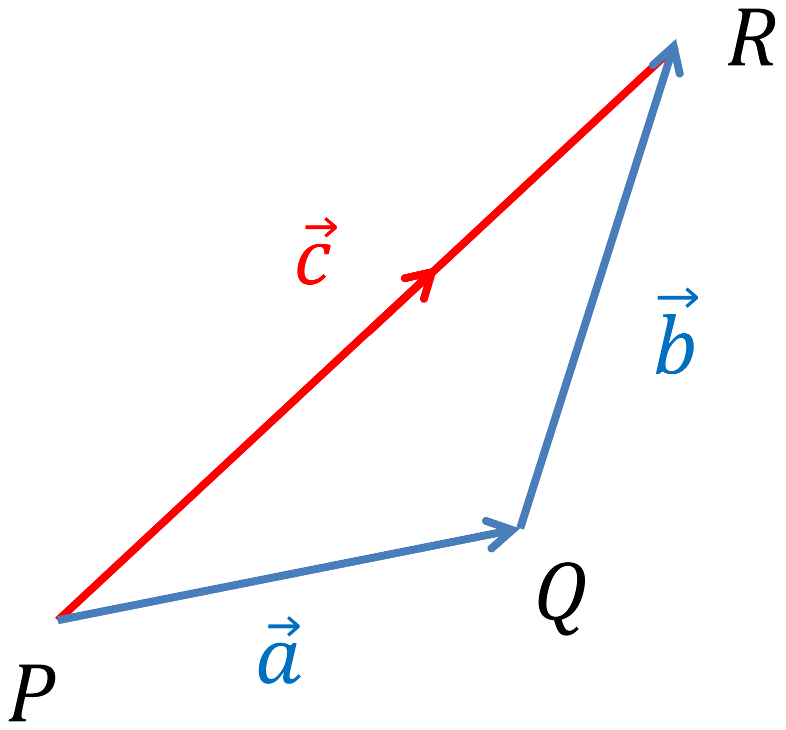

Figure 2

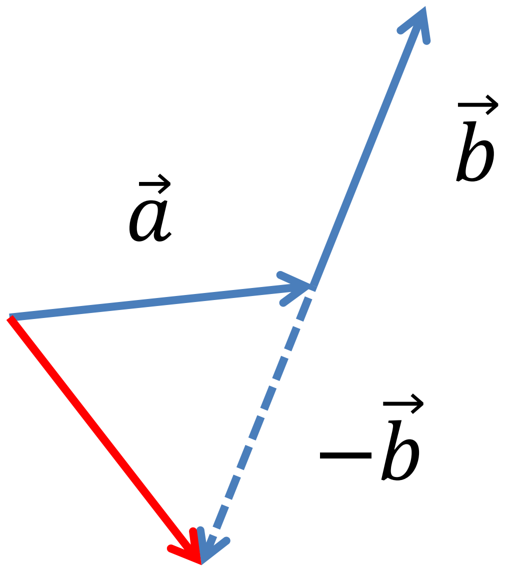

Figure 3

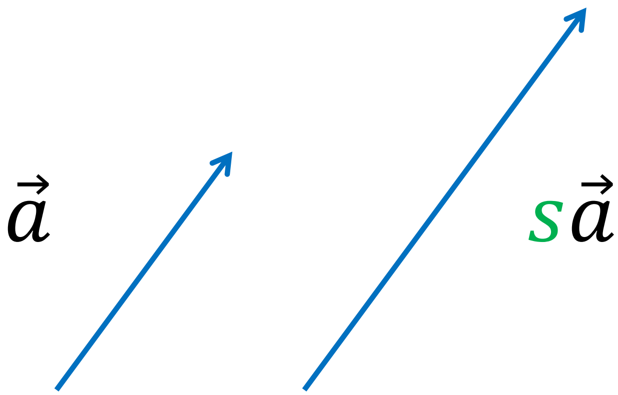

Figure 4

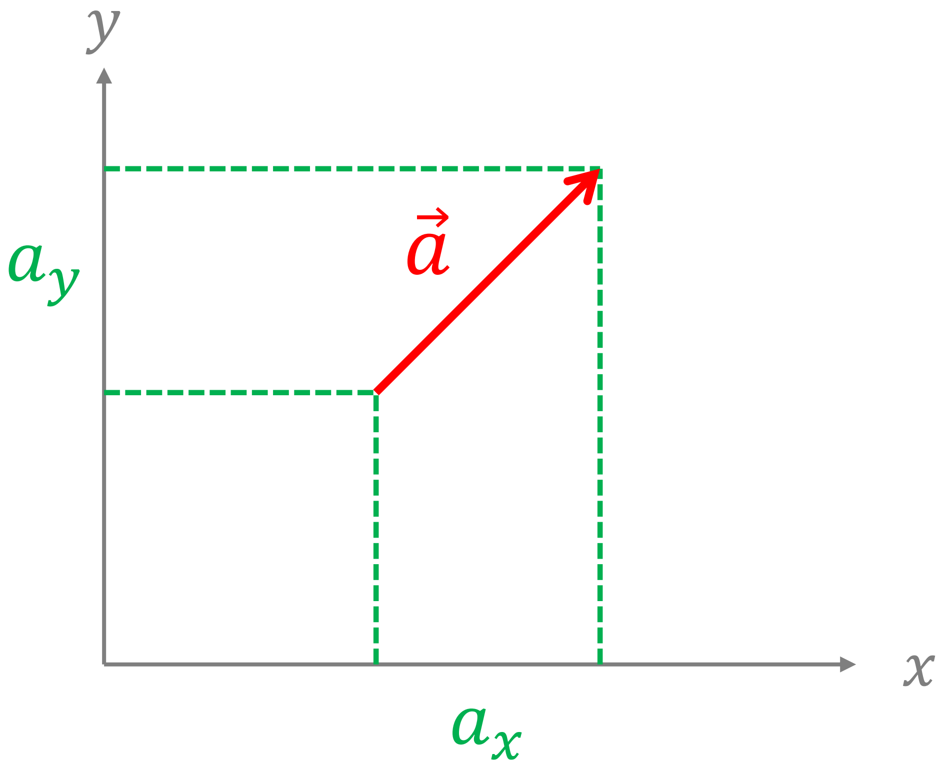

Figure 5

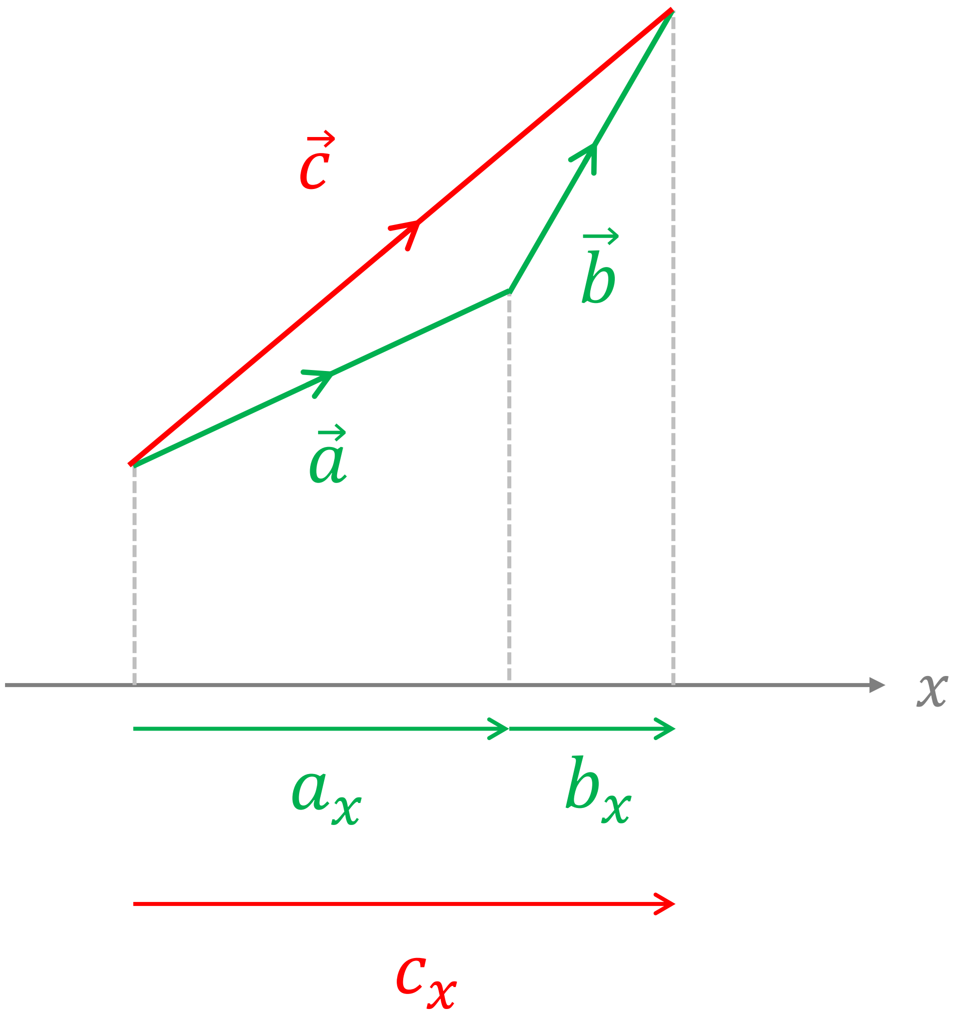

Figure 6

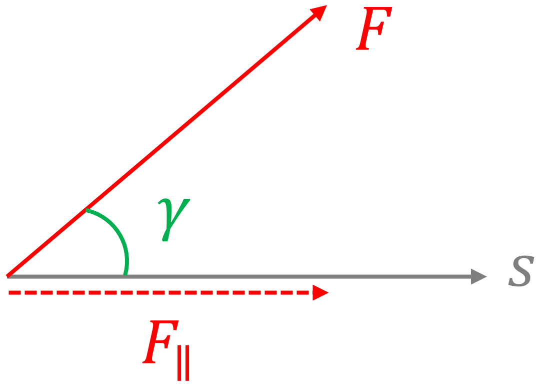

Figure 7

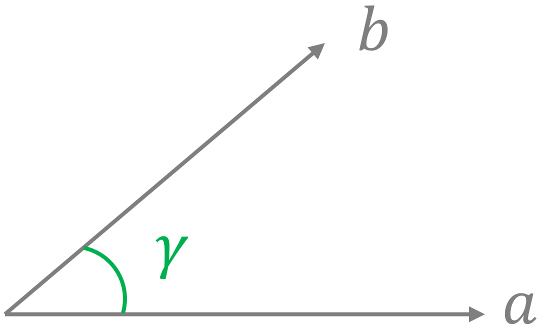

Figure 8

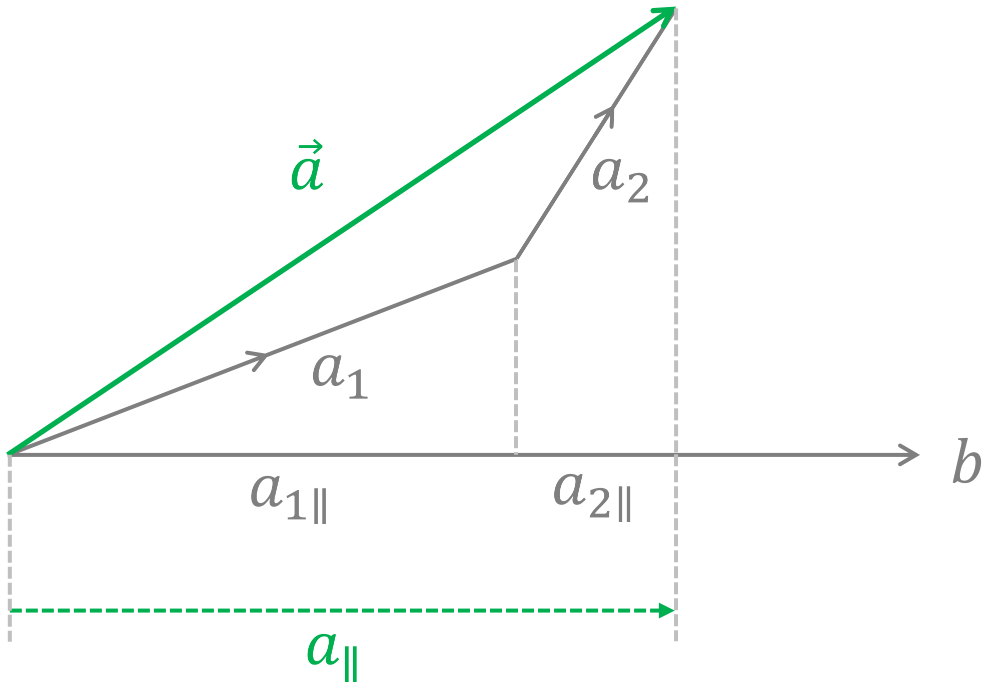

Figure 9

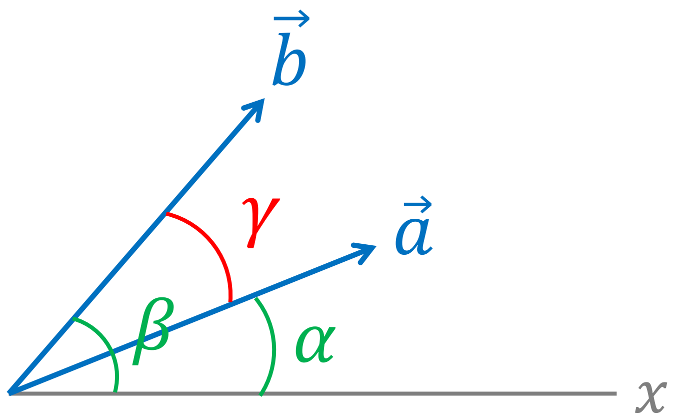

Figure 10

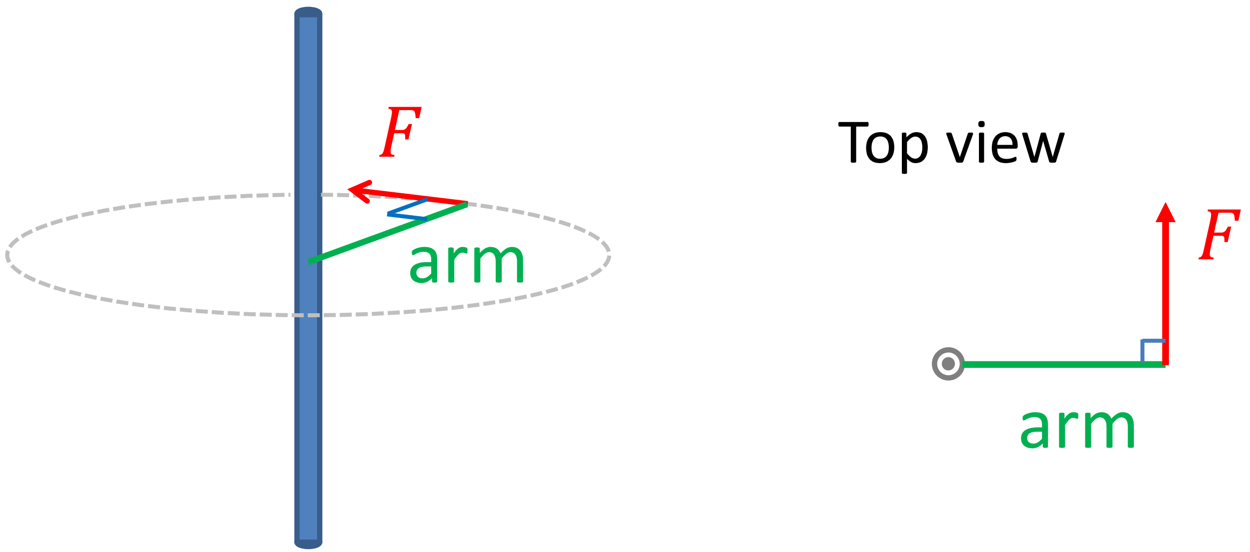

Figure 11a

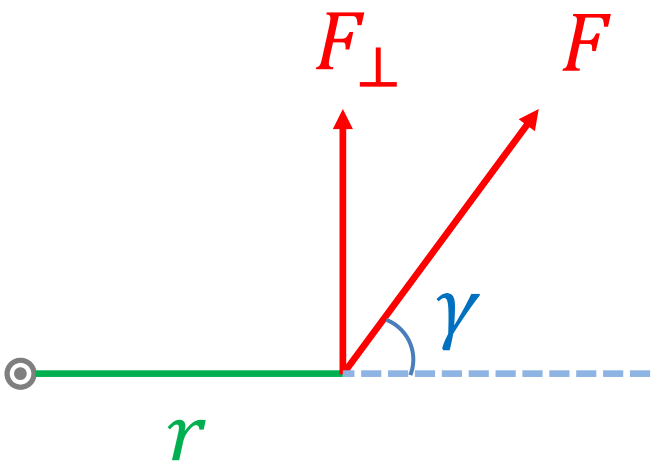

Figure 11b

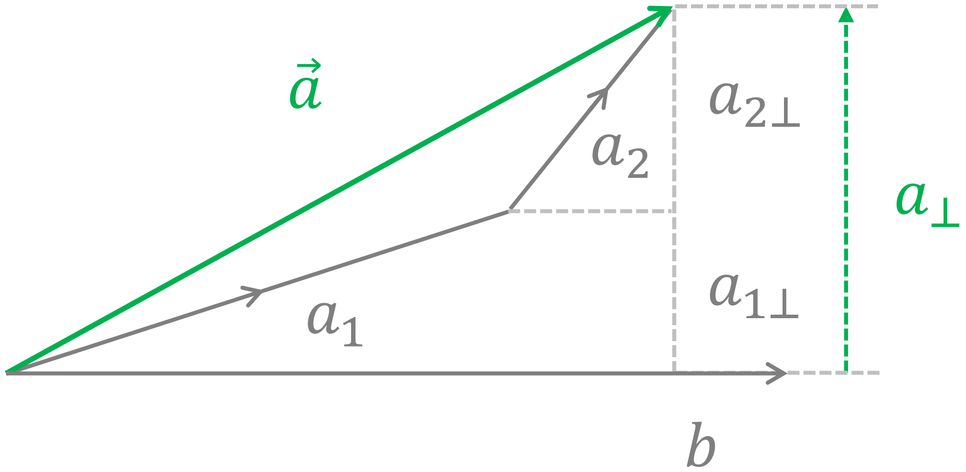

Figure 12

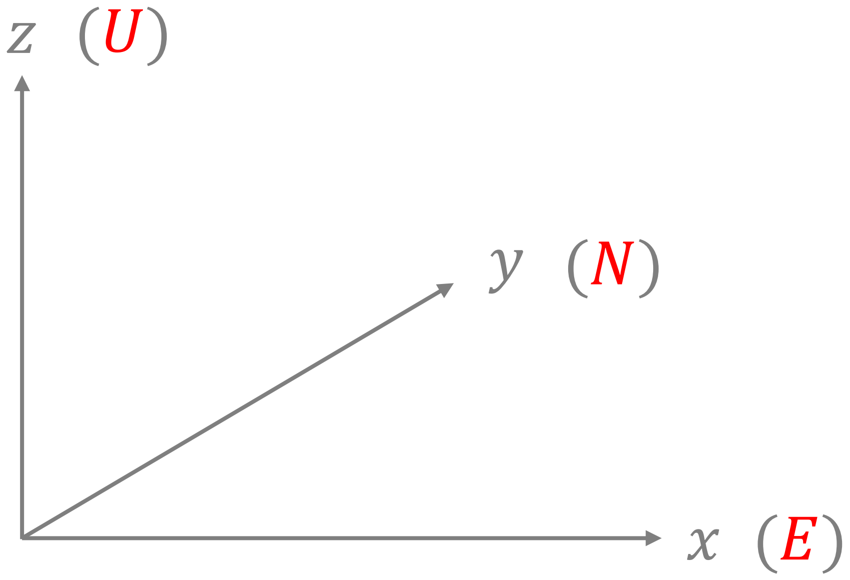

Figure 13

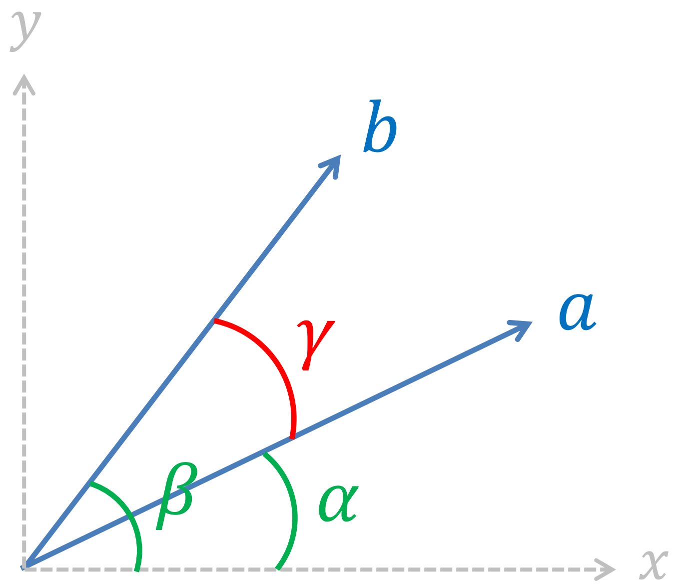

Figure 14

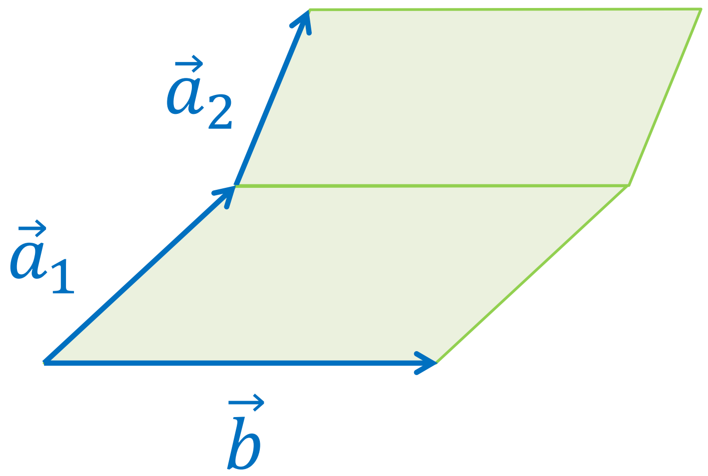

Figure 15a

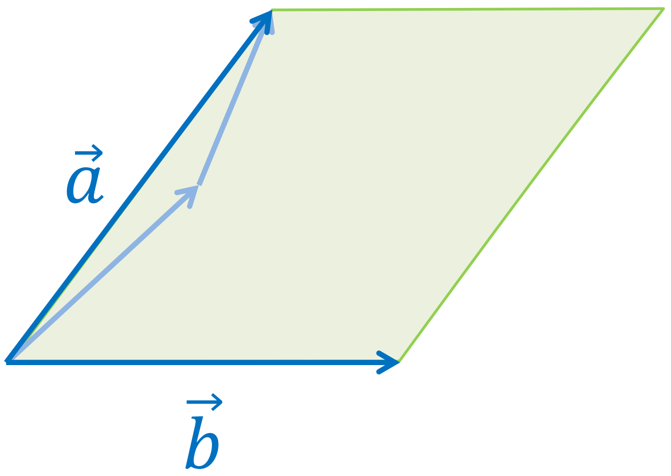

Figure 15b

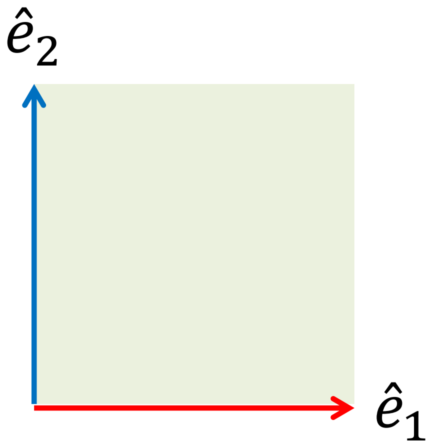

Figure 16a

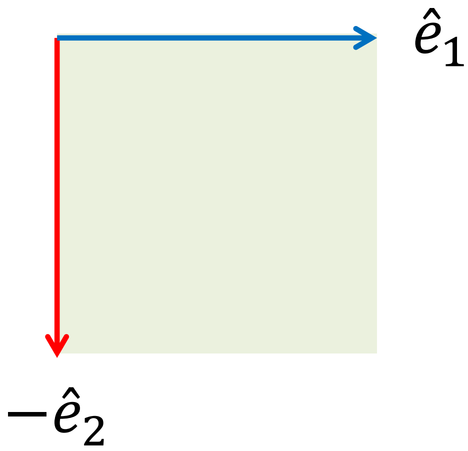

Figure 16b

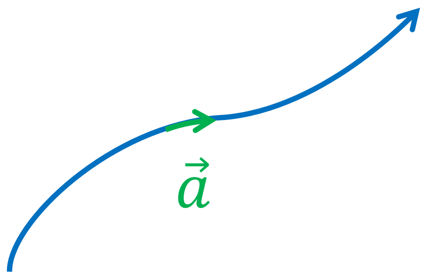

Figure 17a

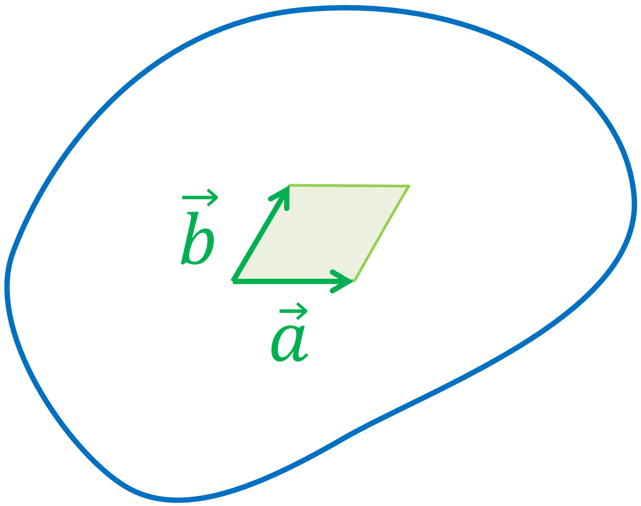

Figure 17b

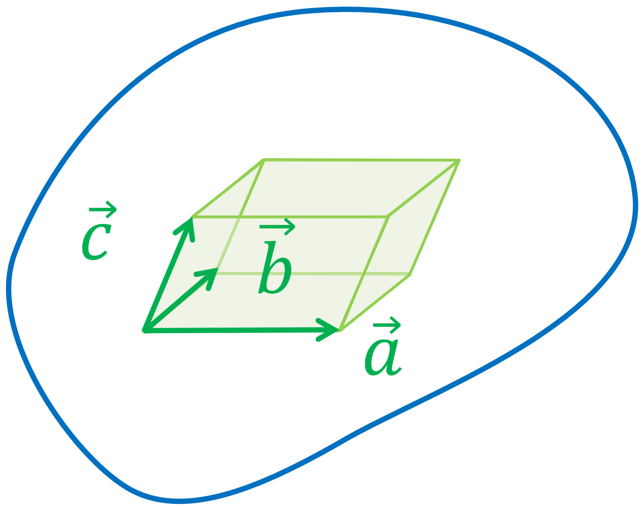

Figure 17c