Coordinates

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Common coordinate systems used in physics are described. The corresponding formulas for distances, areas and volumes are introduced.

Contents

1 Introduction

In simple terms, mechanics is about the motion of point particles, and of bodies (whether or not rigid) made up of point particles. Motion, in turn, is the change of *position* with time. Graphical representation of position becomes inadequate for high accuracy, for computerization, and in higher dimensions. In these circumstances, the position is best represented by *coordinates*, reducing geometry to algebra.

Historically, Newton's work on mechanics in *Philosophiae Naturalis Principia Mathematica* relied a lot on pictures, but that would hardly be adequate for the more complex problems we deal with nowadays.

2 Cartesian coordinates

The systematic use of coordinates to deal with geometry was due to Rene Descartes,[1] who introduced rectangular or Cartesian[2] coordinates.

2.1 Cartesian coordinates on a line

Position of a point

A straight line is labelled as the x axis (**Figure 1a**). An arbitrary point O is chosen as the *origin*. The coordinate x of a point P is the distance OP. The sign convention is that $x > 0$ if P is to the "right" or in a conventionally chosen positive direction. The

[1] In the family name, both occurrences of "s" are silent.

[2] The Latin version of his name, Renatus Cartesius, would be rendered into an adjective this way.

coordinates of different points $P, Q, \ldots$ can be denoted as $x_P, x_Q, \ldots$ or $x_1, x_2, \ldots$ if numerical labels are assigned to the points.

Displacement
Figure 1b shows two points on the line, with coordinates x_1, x_2. These can be two cities along a straight road, or the position of a particle at two times. The *displacement* of one with respect to the other is

$$\boxed{\Delta x = x_2 - x_1} \tag{1}$$

which is a signed quantity: the displacement is positive (negative) if x_2 is to the "right" ("left") of x_1 — we shall often use the shorthand "right" ("left") to mean "in the direction of the positive (negative) x-axis".

Automatic signs
The formula (1) automatically takes care of the sign; it can (and should) be used "blindly". For example, **Figure 2a** shows two points at positions $x_1 = 3$, $x_2 = 7$, with the displacement

$$\Delta x \quad = \quad (7) - (3) = 4$$

Figure 2b shows two points at positions $x_1 = -1$, $x_2 = 3$, and the displacement is

$$\Delta x \quad = \quad (3) - (-1) = 4$$

In both cases the values of x_i are enclosed in brackets.

Translation of origin
In physics, we encounter an interesting problem:

- An origin is needed in order to talk about coordinates.
- But the choice of origin is arbitrary, and physics cannot depend on that choice.

We say that physics should be *invariant* with respect to a shift or translation of origin.

Figure 2a and **Figure 2b** differ only by a shift of origin. Physics can only depend on Δx (but not x_1 and x_2 individually), since only Δx is invariant. The same considerations in higher dimensions will not be repeated.

2.2 Cartesian coordinates on a plane and in space

Plane
For a plane in two dimensions (2D), again choose an *origin* O and construct the x and y axes, which are mutually *perpendicular* going through O. The position of a point P is then specified by the coordinates normally written as a pair (x, y); see **Figure 3a**. We can imagine the plane to be a piece of paper with square grids drawn on it; the two coordinates are then the number of grid steps (respectively "horizontally" and "vertically") to go from O to P.

Space
For space in three dimensions (3D), again choose an *origin* and construct the x, y and z axes, which are mutually *perpendicular* through O . The position of a point P is specified by the coordinates, normally written as a triple (x, y, z); see **Figure 3b**.

In particular, coordinates defined with respect to such mutually perpendicular axes are said to be *rectangular* or *Cartesian.*

Displacement
Given two points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in space, the *displacement* is

$$(\Delta x, \Delta y, \Delta z) \quad = \quad (x_2{-}x_1, y_2{-}y_1, z_2{-}z_1)$$

For 2D simply drop the last entry, or equivalently constrain all z_i to be zero.

Ordered *N*-tuple
In writing the coordinates as an ordered pair or ordered triple, we are anticipating the general definition of a *vector* as an ordered N-tuple, which is deferred to a later module.

Orientation of axes
The "normal" convention is that the axes are chosen such that

- the x axis points "east";
- the y axis points "north";
- the z axis points "up".

The three directions are indicated in quotation marks because they may be merely names, and the directions need not be those encountered in geography — for example if we are considering a point particle in outer space.

Figure 4 illustrates, in the case of 2D, two sets of axes and the corresponding rectangular coordinates (x, y) and (x', y'). The case of 3D can be imagined.

In physics, we are faced with another interesting problem:

- We need a set of axes in order to talk about coordinates.
- But the choice of axes is arbitrary, and physics cannot depend on that choice.

We say that physics should be *invariant* under a rotation of axes.

3 Polar coordinates on a plane

3.1 Definition of polar coordinates

On a plane, a point P can be specified by its distance r from the origin O, and the angle ϕ between OP and the $+x$ axis, as shown in **Figure 5a**.[3] Unless otherwise specified, angles are measured in radians.

The range is

$$\begin{aligned} &0 \le r < \infty \\ &0 \le \phi < 2\pi \quad \text{or} \quad -\pi < \phi \le \pi \end{aligned}$$

The range of ϕ can be shifted by any multiple of 2π.

Such *polar* coordinates can be associated with a curvilinear grid on the plane (**Figure 5b**): circles (linking points of constant r) and half straight lines through the origin (linking points of constant ϕ).

3.2 Relationship with Cartesian coordinates

To relate to Cartesian coordinates, refer to **Figure 6**:

$$\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned} \tag{2}$$

The reverse transformation is

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \phi &= \arctan(y/x) \end{aligned} \tag{3}$$

[3]In anticipation of the polar angle in 3D, the angle on the plane is denoted as ϕ and not θ.

The formula for ϕ, as written above, is not quite enough — there are two solutions for arctan, differing by π. For example, if $y/x = 1$, then $\phi = \pi/4, 5\pi/4$. The former (latter) is chosen if y and x are individually positive (negative).

Problem 1
Find the polar coordinates for a point $(x, y) = (3.0, 4.0)$. §

Problem 2
Find the Cartesian coordinates of a point $(r, \phi) = (2, \pi/6)$. §

Example of use
Polar coordinates are obviously convenient for circular motion. Consider a point P in uniform circular motion on a circle of radius R. Then

$$\begin{aligned} r(t) &= R \\ \phi(t) &= \omega t \end{aligned} \tag{4}$$

Only one coordinate changes with time.

Problem 3
Continue with the above example. Find the x and y coordinates of P as a function of time. Also find the components of velocity v_x, v_y and the components of acceleration a_x, a_y. §

4 Cylindrical coordinates

A point P in three-dimensional space can be described by *cylindrical coordinate* (ρ, ϕ, z), as shown in **Figure 7**.

- The coordinate z is the usual Cartesian coordinate, i.e., the height of P above the x–y plane.
- Project P onto the x–y plane to give Q. The coordinates (ρ, ϕ) are just the polar coordinates of Q on this plane. We use the symbol ρ rather than r, in anticipation of spherical coordinates (next Section).

The relationship of cylindrical coordinates with Cartesian coordinates follows trivially from (2) and (3).

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned} \tag{5}$$

with the reverse transformation

$$\begin{aligned} \rho &= \sqrt{x^2+y^2} \\ \phi &= \arctan(y/x) \\ z &= z \end{aligned} \quad (6)$$

Again, the individual signs of x and y should be used to select the two solutions of arctan.

5 Spherical coordinates

A point P in 3D can be specified by a distance and two angles: (r, θ, ϕ).

5.1 Definition of spherical coordinates

Radius
First, r is the distance to the origin O, i.e., the length of OP.

Polar angle
Next, θ is the angle between OP and the z axis, as shown in **Figure 8a**. The set of all points with the same r and θ is a circle, shown as a broken line in **Figure 8b**. The radius of this circle is

$$\rho = r \sin\theta \quad (7)$$

The z coordinate is evidently

$$z = r \cos\theta \quad (8)$$

Azimuthal angle
The circle of radius ρ formed by the broken line is parallel to the x–y plane; it is shown in **Figure 9**. Call its center O'. The position of P on this circle is specified by an angle ϕ, measured from the x axis to the line $O'P$.

In fact, (ρ, ϕ) constitute polar coordinates on this 2D plane, while (ρ, ϕ, z) would constitute cylindrical coordinates.

Range of values
The allowed ranges are

$$\begin{aligned} &0 \le r < \infty \\ &0 \le \theta \le \pi \\ &0 \le \phi < 2\pi \quad \text{or} \quad -\pi < \phi \le \pi \end{aligned} \quad (9)$$

Note that θ goes only up to π and not 2π.

5.2 Relationship to Cartesian coordinates

Since $x = \rho \cos\phi$, $y = \rho \sin\phi$, from (7), we get

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned} \quad (10)$$

and the reverse transformation is

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \arccos(z/r) \\ \phi &= \arctan(y/x) \end{aligned} \quad (11)$$

As before, the quadrant for ϕ depends on the individual signs of x and y.

5.3 Relationship to latitude and longitude

Suppose r is given. Then P is on the surface of a sphere. Think of the sphere as the earth, and adopt the usual convention that the north pole (NP) lies along the $+z$ axis, the south pole (SP) along the $-z$ axis. Draw latitudes and longitudes on the surface (**Figure 10**), with the 0° longitude going through the x axis. Latitudes North (South) are regarded as positive (negative); longitudes East (West) are regarded as positive (negative). Again note that the latitude has a range of only 180°, whereas the longitude has a range of 360°.

Then it is easily seen that[4]

$$\begin{aligned} \theta &= 90^\circ - \text{latitude} \\ \phi &= \text{longitude} \end{aligned} \quad (12)$$

The relationship with the latitude is illustrated by the table below.

Position	Latitude	θ
NP	$+90^\circ$	0°
Equator	0°	90°
SP	-90°	180°

Table 1. Latitude and polar angle

Problem 4
The radius of the earth is 6370 km, and the positions of Hong Kong (H) and Buenos Aires (B)

[4] Here it is more convenient to use degrees for the angles, since that is the norm for latitudes and longitudes.

are as follows: $H = 22.4\,^\circ$N, $114.1\,^\circ$E; $B = 34.6\,^\circ$S, $58.4\,^\circ$W. (a) Find their respective cartesian coordinates with respect to the center of the earth. (b) Imagine that we want to dig a straight tunnel through the interior of the earth from H to B. How long would this tunnel be? §

6 Formulas for distance

6.1 Cartesian coordinates

Basic formula
Consider neighboring points P and Q on a plane (**Figure 11**) with coordinates (x, y) and $(x+\Delta x, y+\Delta y)$. The distance Δs between them is given by

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$

Generalizing to 3D and taking the separation to be infinitesimal,[5], we have

$$\boxed{ds^2 = dx^2 + dy^2 + dz^2} \tag{13}$$

In such formulas, dx^2 always means $(dx)^2$ and not $d(x^2)$, etc.

Length of a curve
The length of a curve is obviously

$$\begin{aligned} s &= \int ds \\ &= \int \sqrt{dx^2 + dy^2 + dz^2} \end{aligned} \tag{14}$$

But how is such a strange integral to be evaluated? We illustrate through an example.

Consider a particle whose coordinates are given in terms of time t: $x = x(t)$, $y = y(t)$, $z = z(t)$. Then on the RHS of (14) we can divide and multiply by dt:

$$\begin{aligned} s &= \int \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)^{1/2} dt \\ &\equiv \int f(t)\, dt \end{aligned} \tag{15}$$

where $\dot{x} = dx/dt$ etc. This is then an ordinary integral of some function $f(t)$.

[5] This is not really necessary here, but for the other coordinate systems below, the analogous limit is required.

Example 1
The motion of a projectile is given by

$$\begin{aligned} x(t) &= u_0 t \\ y(t) &= 0 \\ z(t) &= v_0 t - (1/2) g t^2 \end{aligned}$$

We have $\dot{x} = u_0$, $\dot{y} = 0$, $\dot{z} = v_0 - gt$, so

$$s = \int \left[u_0^2 t^2 + (v_0 - gt)^2\right]^{1/2} dt$$

The limits are the initial time t_1 and the final time t_2. §

The point is simply that the strange-looking object (14) can be converted to an ordinary integral; the evaluation of the latter is a separate matter of ordinary calculus. Incidentally, one can label the position by any parameter say σ, e.g., $x = x(\sigma)$, and divide and multiply by $d\sigma$, so that (15) becomes

$$\begin{aligned} s &= \int \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)^{1/2} d\sigma \\ &\equiv \int f(\sigma)\, d\sigma \end{aligned} \tag{16}$$

where now $\dot{x} = dx/d\sigma$ etc.

Problem 5
Calculate the circumference of a circle, given by $x = R\cos\sigma$, $y = R\sin\sigma$, $0 \le \sigma < 2\pi$. §

6.2 Polar coordinates on a plane

Basic formula
Consider two neighboring points P and Q on a plane (**Figure 12**) with polar coordinates (r, ϕ) and $(r+\Delta r, \phi+\Delta\phi)$; the quantities Δr and $\Delta\phi$ should be regarded as infinitesimal. Refer to the two line segments shown in **Figure 12**. The radial separation is Δr and the tangential separation is $r\Delta\phi$. Moreover these line segments are perpendicular. So the distance Δs between P and Q is given by

$$(\Delta s)^2 = (\Delta r)^2 + (r\Delta\phi)^2$$

Passing to differentials,

$$\boxed{ds^2 = dr^2 + r^2\, d\phi^2} \tag{17}$$

Algebraic derivation
It is instructive to derive (17) algebraically, without appealing to the diagram in **Figure 12**. The algebraic method is more generally applicable to more complicated coordinate systems.

Start with the Cartesian coordinates on a plane using (2), and consider their differentials:

$$\begin{aligned} dx &= \cos\phi\, dr - r\sin\phi\, d\phi \\ dy &= \sin\phi\, dr + r\cos\phi\, d\phi \end{aligned}$$

Square these and put into (17). The expression will be quadratic in the differentials, i.e., containing terms of three types

$$dr^2, \quad dr\, d\phi, \quad d\phi^2$$

Collect these terms and simplify.

Problem 6
Carry out the above steps and prove (17). Note in particular that the cross terms cancel. (This is not guaranteed in general.) §

Problem 7
Derive the distance formula for cylindrical coordinates (ρ, ϕ, z). §

Problem 8
A spiral is given by

$$\begin{aligned} \rho(t) &= R \\ \phi(t) &= 2\pi t \\ z(t) &= pt \end{aligned}$$

in terms of a parameter t. There is one turn of the spiral for each unit increase in t and p is the pitch. Find the length of this spiral per turn. It is sufficient to reduce the expression to an ordinary integral; you do not have to evaluate this integral. §

6.3 Spherical coordinates

Preliminaries
Some examples will help us to understand the general formula.

Problem 9
The radius of the Earth is = r = 6370 km. Hong Kong is at H = 22.3023 °N, 114.1741 °E.[6]

[6]These are the coordinates of Hong Kong Observatory

(a) If a point P is slightly due south, at P = 22.3024 °N, 114.1741 °E, what is the distance HP? Generalize this to the north-south distance between two points separated by $d\theta$ in polar angle.
(b) If a point Q is slightly due east, at Q = 22.3023 °N, 114.1742 °E, what is the distance HQ? Hint: What is the radius ρ of the line of latitude? Generalize this to the east-west distance between two points separated by $d\phi$ in azimuthal angle. §

Basic formula
Consider two neighboring points P and Q in 3D with polar coordinates (r, θ, ϕ) and $(r+\Delta r, \theta+\Delta\theta, \phi+\Delta\phi)$; the quantities Δr, $\Delta\theta$ and $\Delta\phi$ should be regarded as infinitesimal. For simplicity **Figure 13** shows the case $\Delta r = 0$; you should imagine that Q lies on a slightly larger sphere. It may be useful to think about the two points as near the surface of the earth, with θ and ϕ related to latitude and longitude in the usual way. The vector separation from P to Q has three components (immediately passing to differentials):

- In the east-west direction, with a length $\rho\, d\phi = r\sin\theta\, d\phi$. Here ρ is the radius of the line of latitude.
- In the north-south direction, with a length $r\, d\theta$.
- In the up-down direction, with a length dr.

The three separations are mutually perpendicular, so

$$\boxed{ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2} \qquad (18)$$

Problem 10
Derive the above formula algebraically, following the method in Section 6.2. §

7 Formulas for area and volume

7.1 Cartesian coordinates

Area on a plane
Areas on a plane can be regarded as made up of small rectangles, of dimension $\Delta x \times \Delta y$. Passing

to the infinitesimal limit, the element of area is[7]

$$\boxed{dS = dx\,dy} \qquad (19)$$

While surface integrals will not be dealt with systematically, one example will illustrate how areas are calculated.

Example 2
The semicircular disk is bounded by the x axis and the semi-circle $y = \sqrt{R^2 - x^2}$ (**Figure 14a**). To calculate the area of the disk, we simply write

$$S = \int dS \quad = \quad \iint dx\,dy \qquad (20)$$

The *surface integral* has been written as a *double integral*, which can be understood as follows. The total area is the sum of little rectangular elements, which we add up in two steps. First, for a fixed x, we add up the elements for different y (i.e., a "column" as in **Figure 14b**). The limits for y are $y = 0$ and $y = \sqrt{R^2 - x^2}$. Then we add up all the "columns" labelled by x, which ranges from $-R$ to R. Thus

$$S \quad = \quad \int_{-R}^{R} dx \int_0^{\sqrt{R^2-x^2}} dy \qquad (21)$$

It is often convenient to write multiple integrals this way: the differentials not at the end, but each right next to the corresponding integral sign. The meaning is always to do the inner integral first.

In this case, the inner integral is simple

$$\int_0^{\sqrt{R^2-x^2}} dy \quad = \quad \sqrt{R^2 - x^2}$$

so

$$S \quad = \quad \int_{-R}^{R} dx\,\sqrt{R^2 - x^2} = \pi R^2/2$$

which is of course correct. §

Problem 11
Redo the above problem, but adding up the "rows" first, i.e., first integrate over x, then integrate over y. Pay attention to the limits of the integrals. §

Volume in 3D space
Volumes in 3D space can be regarded as made up of small rectangular blocks, of dimension $\Delta x \times \Delta y \times \Delta z$. Passing to the infinitesimal limit, the element of volume is

$$\boxed{dV = dx\,dy\,dz} \qquad (22)$$

Example 3
A hemisphere is bounded by the x–y plane and the surface $z = \sqrt{R^2 - x^2 - y^2}$. To calculate the volume of the hemisphere, we write

$$V = \int dV \quad = \quad \iiint dx\,dy\,dz \qquad (23)$$

The *volume* integral has been written as a *triple integral*. Again we do it starting from the inner integrals.

The calculation can be finessed a little by recognizing that, at fixed z,

$$\begin{aligned} & \iint dx\,dy \\ = \;& \text{area of circle at height } z \\ = \;& \pi\,\rho^2 = \pi\,(R^2 - z^2) \end{aligned}$$

Hence we are left with the last integral, giving

$$\begin{aligned} V \quad &= \quad \int_0^R \left[\pi\,(R^2 - z^2)\right]\,dz \\ &= \quad (2\pi/3)R^3 \end{aligned}$$

which is of course the right answer. §

7.2 Polar coordinates on a plane

Basic formula
Areas on a plane can also be regarded as made up of small elements bounded by neighboring lines of constant radius (r and $r+\Delta r$) and lines of constant angle (ϕ and $\phi+\Delta\phi$); see **Figure 15**. So long as Δr and $\Delta\phi$ are small, the area is nearly rectangular, with sides Δr and $r\,\Delta\phi$, and hence area

$$\Delta r \cdot r\,\Delta\phi$$

Passing to infinitesimal elements, we can write the element of area as

$$\boxed{dS = r\,dr\,d\phi} \qquad (24)$$

Example 4
We can redo Example 2 very easily using polar coordinates:

$$S \quad = \quad \iint r\,dr\,d\phi$$

[7]Often the symbol S is used to denote surface area.

Now the limits are easy: r from 0 to R, and ϕ from 0 to π; the limits do not connect the two variables, and the double integral becomes the product of two single integrals:

$$\begin{aligned} S &= \int_0^R dr\, r \cdot \int_0^\pi d\phi \\ &= (R^2/2) \cdot \pi = \pi R^2/2 \end{aligned}$$

agreeing with the previous result. §

Integral of a gaussian

The formula for area provides a neat way of evaluating the integral

$$I \equiv \int_{-\infty}^{\infty} dx\, e^{-x^2}$$

Let us consider

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} dx\, e^{-x^2} \cdot \int_{-\infty}^{\infty} dy\, e^{-y^2} \\ &= \iint dx\, dy\, e^{-(x^2+y^2)} \end{aligned}$$

Regard x and y as Cartesian coordinates and change to polar coordinates:

$$\begin{aligned} dx\, dy &= r\, dr\, d\phi \\ x^2 + y^2 &= r^2 \end{aligned}$$

giving

$$\begin{aligned} I^2 &= \iint r\, dr\, d\phi\, e^{-r^2} \\ &= \int_0^{2\pi} d\phi \cdot \int_0^\infty dr\, r\, e^{-r^2} \\ &= 2\pi \cdot (1/2) = \pi \end{aligned}$$

upon changing variables to $u = r^2$, $du = 2r\, dr$.

Since $I > 0$, we finally get

$$I = \sqrt{\pi} \tag{25}$$

This integral cannot be done by the usual "elementary" means.

Volume in cylindrical coordinates

In cylindrical coordinates (ρ, ϕ, z), an element of volume can be represented as a slab: an area projected on the plane given by (24) (but with r replaced by ρ), with a thickness Δz. It is then obvious that

$$\boxed{dV = \rho\, d\rho\, d\phi\, dz} \tag{26}$$

7.3 Spherical coordinates

Area on a spherical surface

Consider a sphere of radius R. Points on the surface are labelled by the angles θ, ϕ as usual. (Thus, the surface is the set of points (r, θ, ϕ) with r constrained to the constant value R.) Areas on this surface can be regarded as made up of elements bounded by lines of "latitude", i.e., lines of constant θ (at θ and $\theta{+}\Delta\theta$) and lines of "longitude", i.e., lines of constant ϕ (at ϕ and $\phi{+}\Delta\phi$); see **Figure 16a**. So long as $\Delta\theta$ and $\Delta\phi$ are small, this element is a rectangle, with sides $R\,\Delta\theta$ in the "north-south" direction, and $R \sin\theta\, \Delta\phi$ in the "east-west" direction (**Figure 16b**) — remember the circle formed by the latitude has radius $\rho = R\,\sin\theta$. Passing to infinitesimals, the element of surface can be written as

$$\boxed{dS = R^2\, \sin\theta\, d\theta\, d\phi} \tag{27}$$

Problem 12

The radius of the earth is 6370 km. Calculate the area of land (in units of km^2) enclosed by the following lines: the latitudes 22.3024 °N, 22.3024 °N; the longitudes 114.1741 °E, 114.1742 °E. §

Problem 13

Find the surface area of a unit sphere by carrying out the integral

$$S = \iint \sin\theta\, d\theta\, d\phi$$

Put in the appropriate limits. §

Volume in spherical coordinates

A volume in 3D space can be regarded as being made up of elements, each of which is a slab: bounded between two spherical surfaces with radii r and $r{+}\Delta r$, and the area on the spherical surfaces bounded by lines of constant θ and ϕ as above. Then the element of volume is just the surface area in (27), replacing the constant R by the coordinate r, multiplied by the thickness of the slab Δr. Therefore

$$\boxed{dV = r^2\, \sin\theta\, dr\, d\theta\, d\phi} \tag{28}$$

Problem 14

Find the volume of a unit sphere by carrying out the integral of (28) over the appropriate domain. §

Appendix

A Cartesian coordinates in higher dimensions

In space of N dimensions, Cartesian coordinates are formed by an N-tuple of numbers

$$(x_1, x_2, \ldots, x_N)$$

Distances are given by

$$\begin{aligned} ds^2 &= dx_1^2 + dx_2^2 + \ldots + dx_N^2 \\ &= \sum_i dx_i^2 \end{aligned} \tag{29}$$

The element of (hyper)-volume is given by

$$dV = dx_1\, dx_2\, \ldots\, dx_N = \prod_i dx_i$$

B Spherical coordinates in higher dimensions

In N dimensions, spherical coordinates consist of a radius r and $N-1$ angles, which are $N-2$ polar angles plus one azimuthal angle. It would be difficult to present these graphically. Rather we reformulate the analysis in Section 5.2 algebraically, and then generalize to higher dimensions.

Review 3D case
Start with Cartesian coordinates (x, y, z), and define the radius in the obvious way

$$r^2 = x^2 + y^2 + z^2 \tag{30}$$

Of course we take $r > 0$.

Now break the three terms up into two pieces:

$$r^2 = \rho^2 + z^2 \tag{31}$$
$$\rho^2 = x^2 + y^2 \tag{32}$$

and of course we take $\rho > 0$. The first of these motivates the definition

$$z = r\,\cos\theta \quad , \quad \rho = r\,\sin\theta \tag{33}$$

while the second of these motivates the definition

$$x = \rho\,\cos\phi \quad , \quad y = \rho\,\sin\phi \tag{34}$$

However, there is a subtle difference in the introduction of the two angles. In (34), each of x and y can be either positive or negative, so there are four possibilities, corresponding to four quadrants for ϕ; therefore $0 \le \phi < 2\pi$. But in (33), ρ must be positive (by convention), so there are only two choices, corresponding to the first two quadrants for θ; therefore $0 \le \theta \le \pi$.

It is then straightforward to obtain the relation (10). It is also easy to generalize to higher dimensions; only the case of 4D space will be shown.

Spherical coordinates in 4D
In 4D, let the Cartesian coordinates be (x, y, z, w). Define a radius in the obvious way

$$r^2 = x^2 + y^2 + z^2 + w^2 \tag{35}$$

First break this up into two pieces as follows:

$$r^2 = p^2 + w^2 \tag{36}$$
$$p^2 = x^2 + y^2 + z^2 \tag{37}$$

where $p \ge 0$ by convention. The first of these motivates an angle χ:

$$p = r\,\sin\chi \quad , \quad w = r\,\cos\chi \tag{38}$$

By the same reasoning as before, $0 \le \chi \le \pi$.

Next we can break up p^2 by introducing two angles θ and ϕ as is done in normal 3D space, with $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$.

Thus the Cartesian coordinates are given by

$$\begin{aligned} x &= p\,\sin\theta\,\cos\phi \\ &= r\,\sin\chi\,\sin\theta\,\cos\phi \\ y &= p\,\sin\theta\,\sin\phi \\ &= r\,\sin\chi\,\sin\theta\,\sin\phi \\ z &= p\,\cos\theta \\ &= r\,\sin\chi\,\cos\theta \\ w &= r\,\cos\chi \end{aligned} \tag{39}$$

C Minkowski space

The higher dimensional spaces discussed above are called *Euclidean* spaces. They have the property that distances are given by (29), which is the obvious generalization of Pythagoras' theorem, importantly with all terms contributing with + signs. The distance is unchanged when axes are rotated.

But when we consider special relativity, each *event* is defined by four coordinates (t, x, y, z). For convenience, choose the speed of light to be $c = 1$, then the four coordinates have the same units. Then it turns out that the appropriate "distance" formula is

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \qquad (40)$$

with a minus sign for the term associated with time.[8] This formula is "appropriate" in the sense that it is unchanged under Lorentz transformations, i.e., transformations to a moving frame.

A 4D space with one minus sign in the "distance" formula is called *Minkowski* space.

D General coordinates

Define the Cartesian coordinates in 3D space as $\bar{x}_i$, $i = 1, 2, 3$. We can instead use any other three variables x_i, $i = 1, 2, 3$; for example, these can be the spherical coordinates

$$(x_1, x_2, x_3) = (r, \theta, \phi)$$

These coordinates need not have the dimension of length.

Thus in general

$$\begin{aligned} x_1 &= x_1(\bar{x}_1, \bar{x}_2, \bar{x}_3) \\ x_2 &= x_2(\bar{x}_1, \bar{x}_2, \bar{x}_3) \\ x_3 &= x_3(\bar{x}_1, \bar{x}_2, \bar{x}_3) \end{aligned}$$

Any three new variables (subject to some uniqueness and non-singular conditions) constitute a set of *generalized* coordinates.

It is often of interest to express distances, areas and volumes using generalized coordinates; we have done so for several special cases.

Generalized coordinates are useful in several ways.

- The system may have a geometry that is conveniently expressed in terms of these coordinates. For example, circular motion is obviously best expressed in terms of polar coordinates on a plane.
- Suppose a particle is constrained to move on a 2D surface, in general curved. Such a surface can be conveniently expressed by choosing a suitable set of generalized coordinates and setting, for example, $x_3 = C =$ constant. There are no 2D Cartesian coordinates on a curved surface.
- In general relativity, spacetime is curved. Therefore spacetime can be regarded as a 4D surface in flat N dimensional space (with $N > 4$), in which $N - 4$ coordinates have been set to constants.[9] This then provides a set of tools for describing curved spacetime — in which Cartesian coordinates (with either + or − signs in the "distance" formula) are not possible.

[8] Some authors take ds^2 to be the negative of this expression.

[9] Such an embedding approach regards physical spacetime as a subset of a higher-dimensional flat space. It is not immediately obvious that physics would be independent of the extra fictitious dimensions introduced. Therefore it is often preferred, especially by mathematicians, to formulate everything intrinsically, without going beyond the original dimensions. That more abstract approach is probably not suited for the first introduction to curved spacetime.

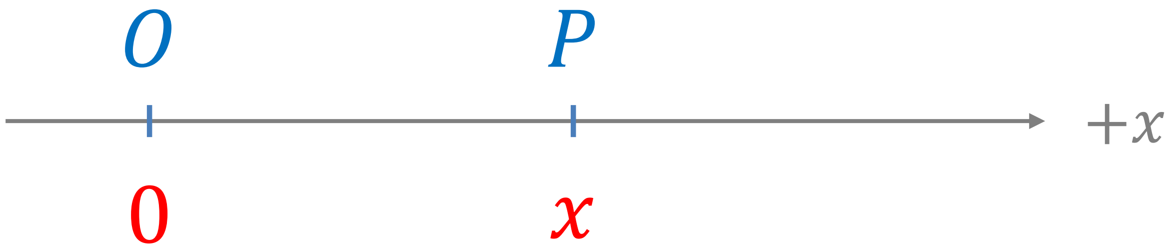


Figure 1a

$$\Delta x = x_2 - x_1$$

Figure 1b

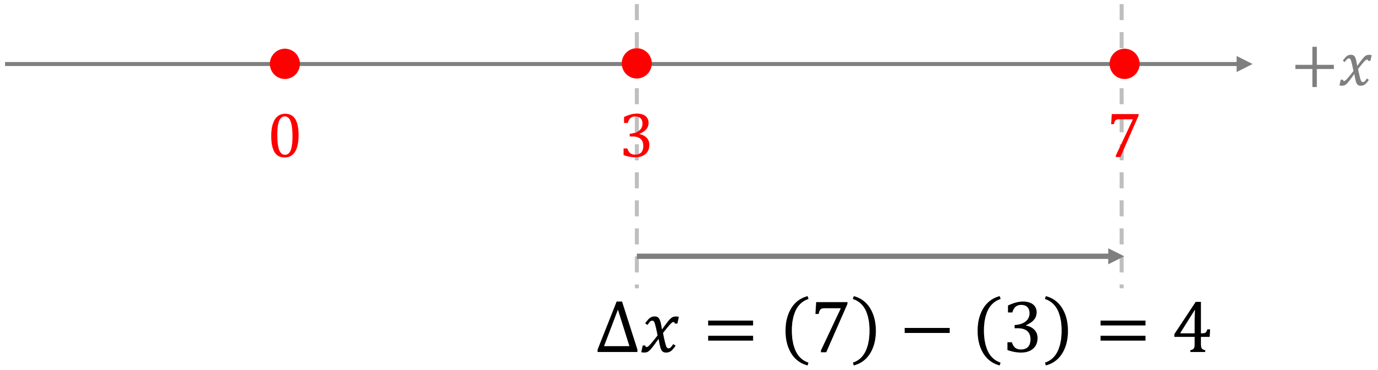


Figure 2a

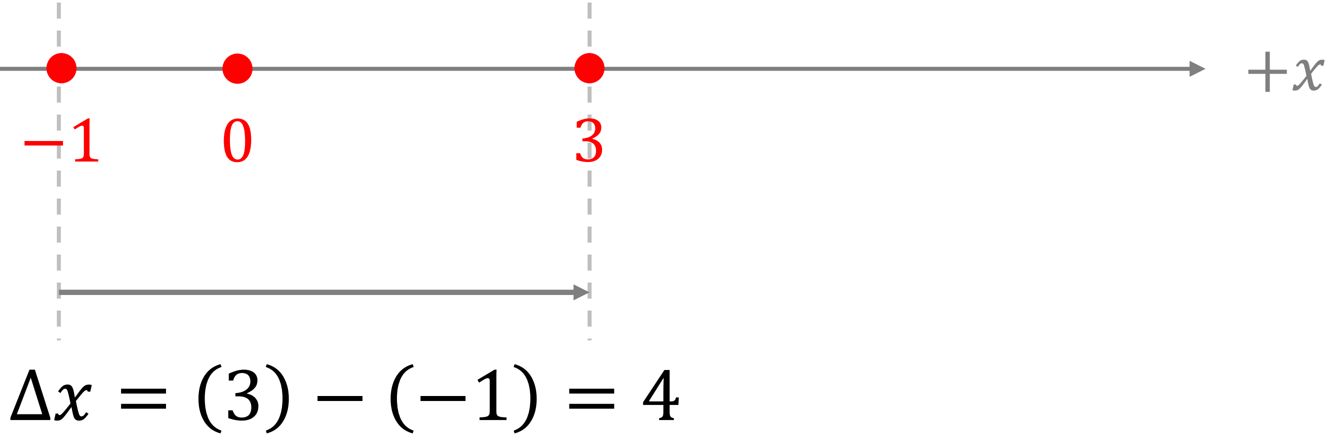


Figure 2b

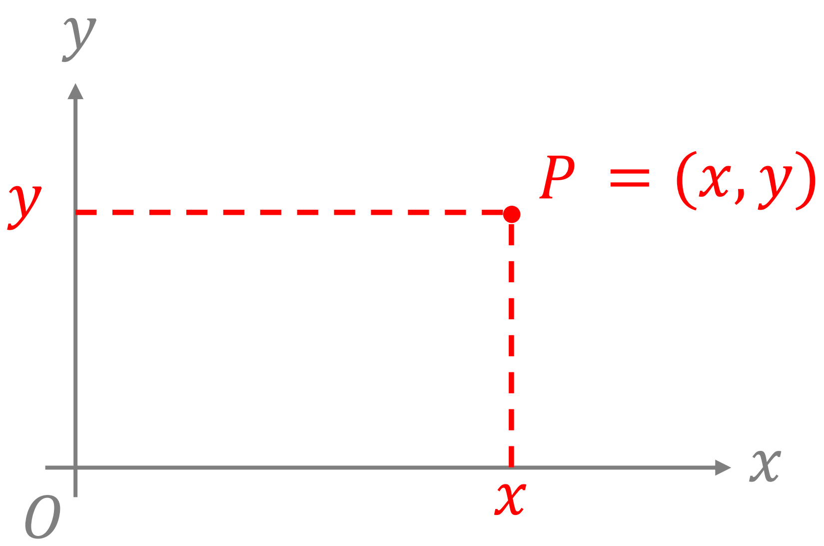


Figure 3a

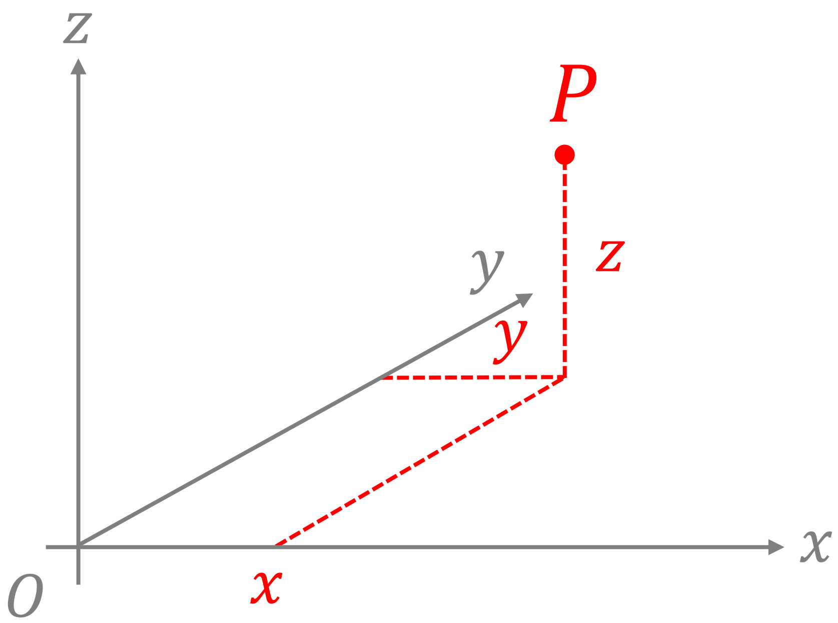


Figure 3b

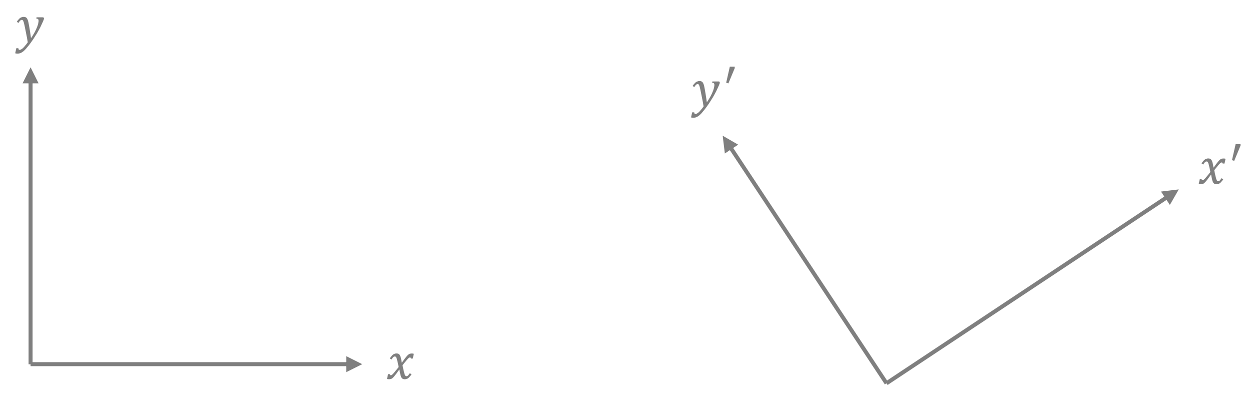
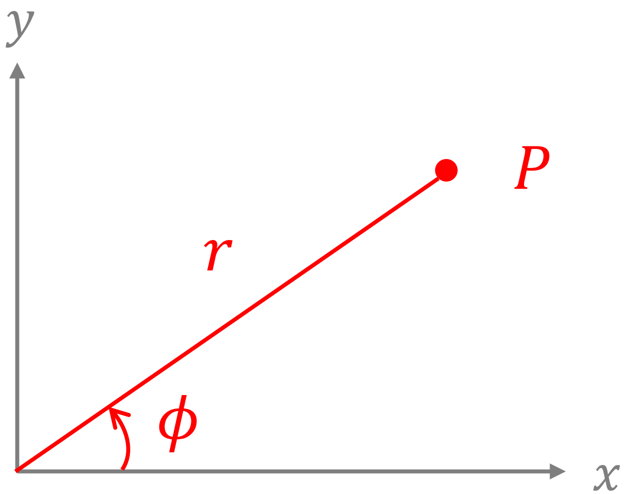
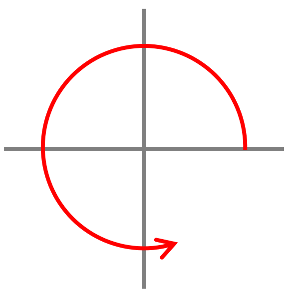


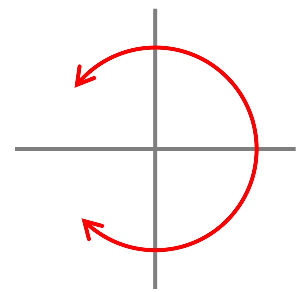
Figure 4



$$0 \leq r < \infty$$



or



$$0 \leq \phi < 2\pi$$

$$-\pi \leq \phi < \pi$$

Figure 5a

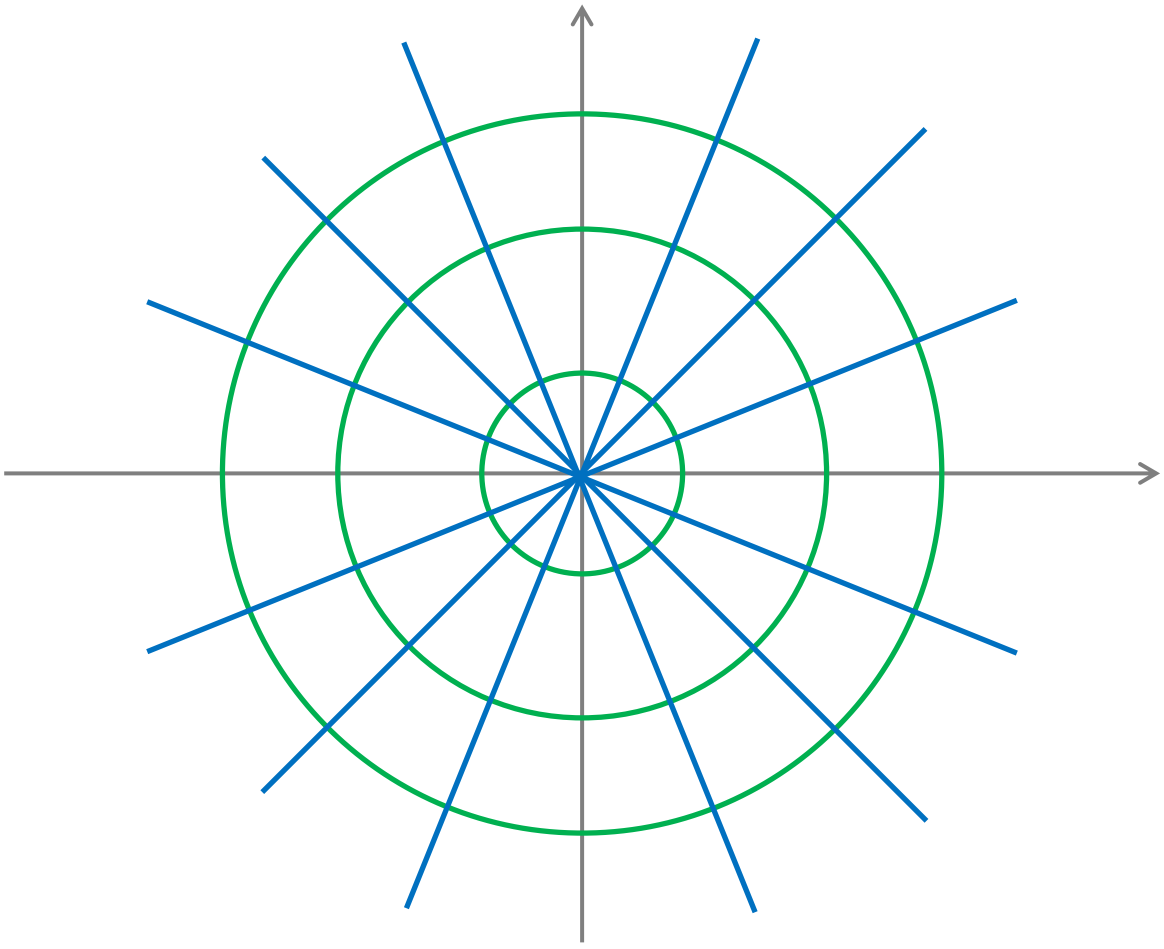


Figure 5b

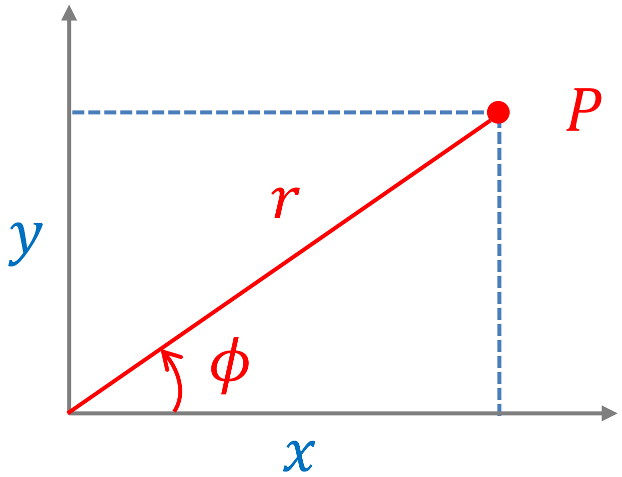


Figure 6

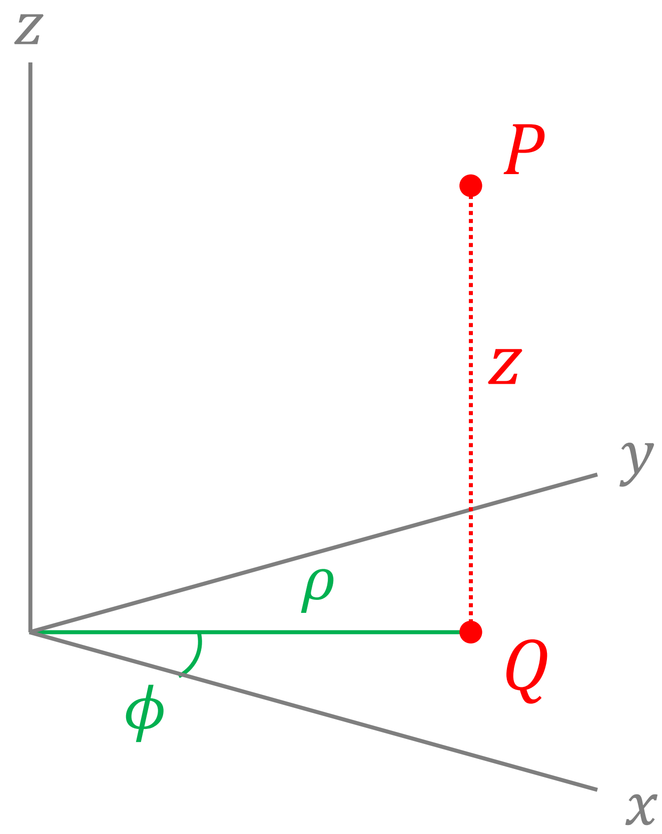


Figure 7

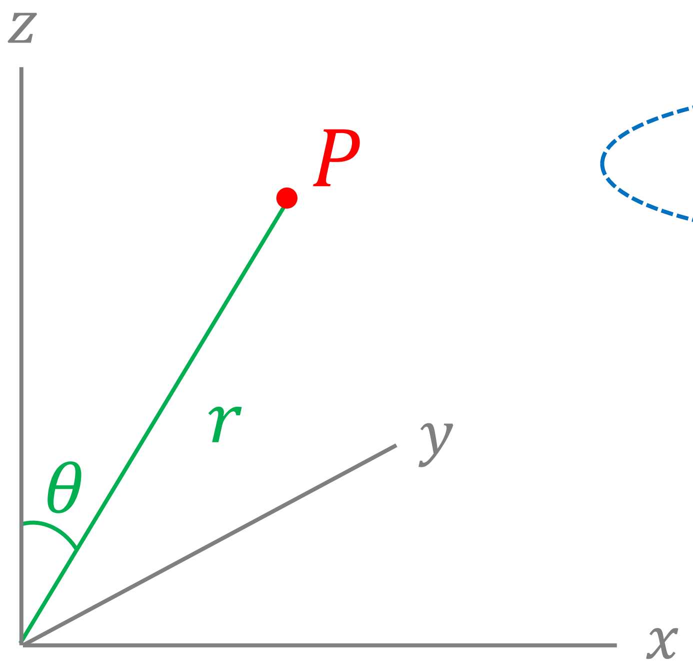


Figure 8a

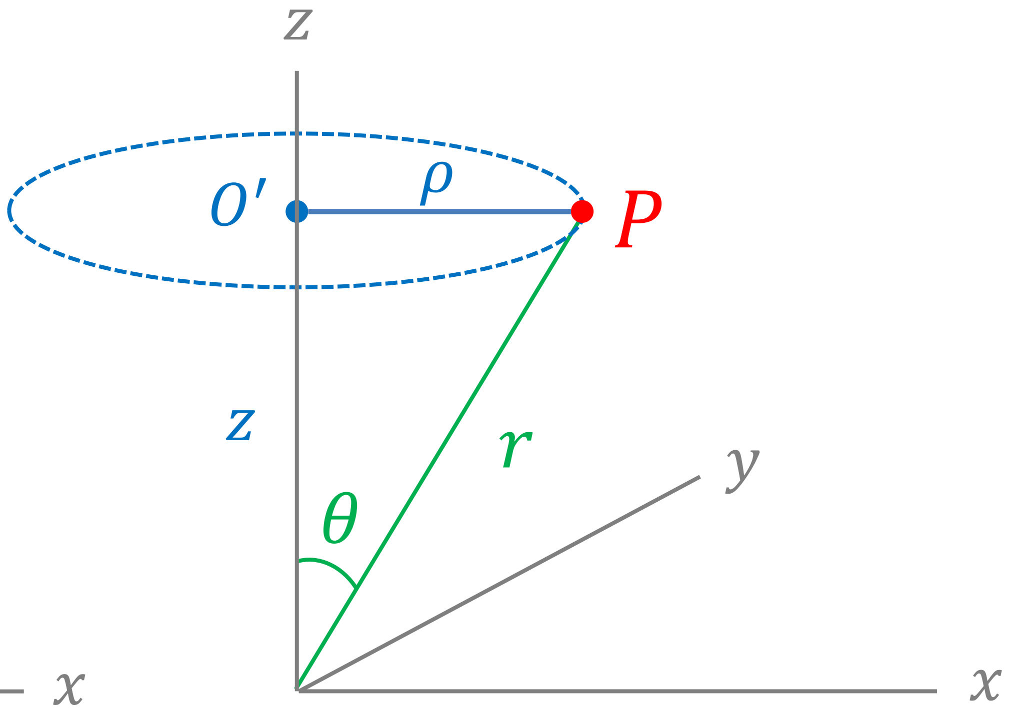


Figure 8b

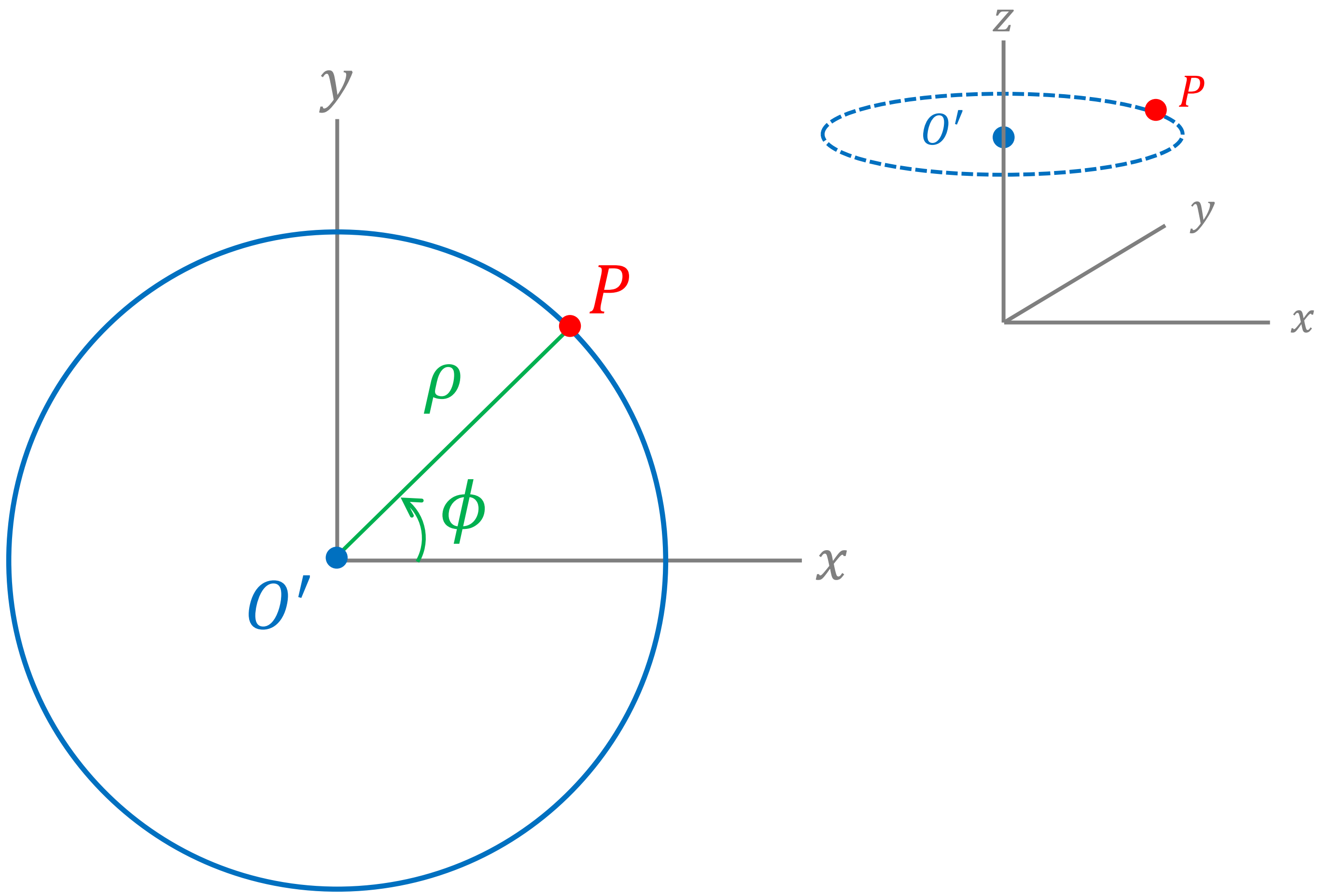


Figure 9

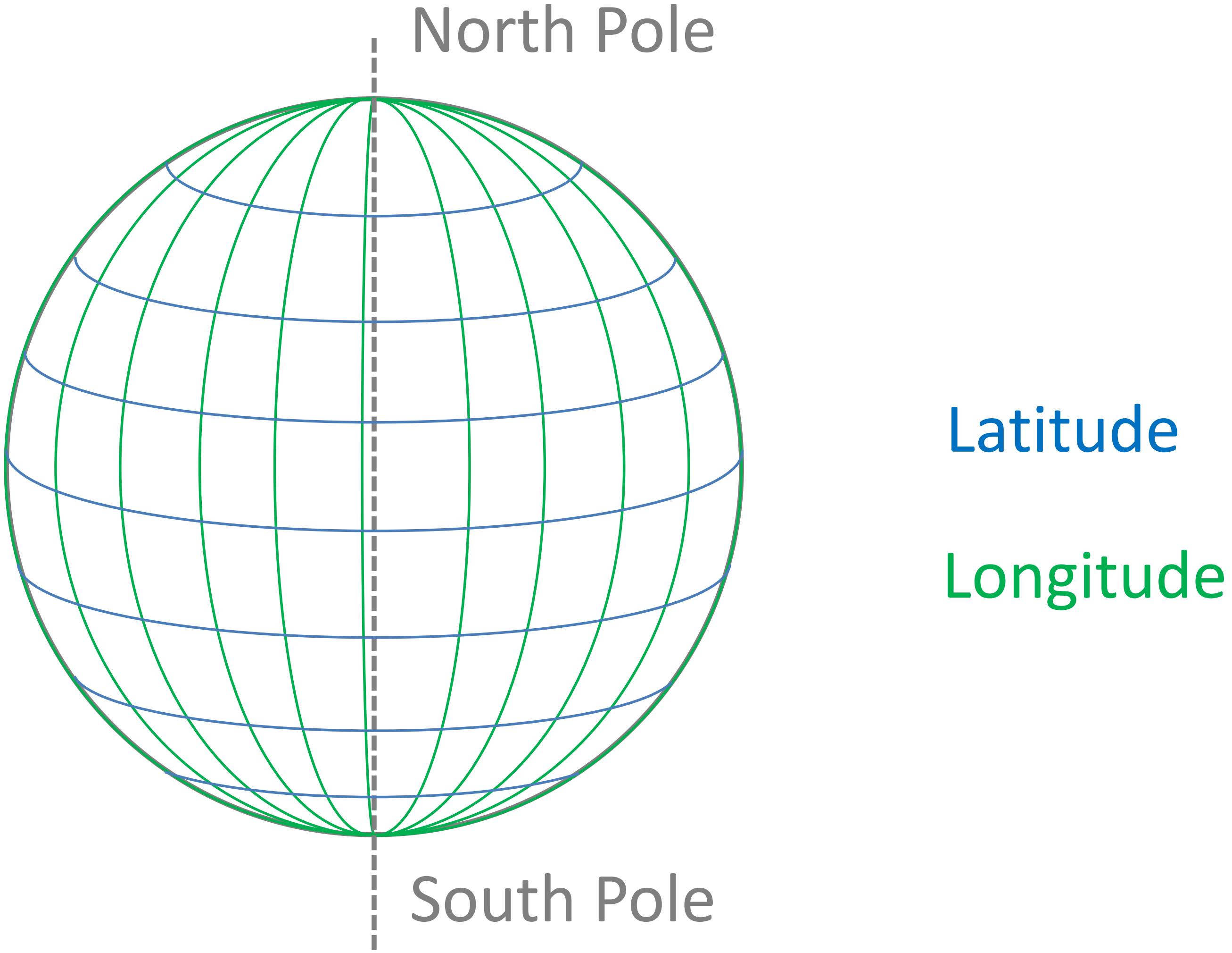


Figure 10

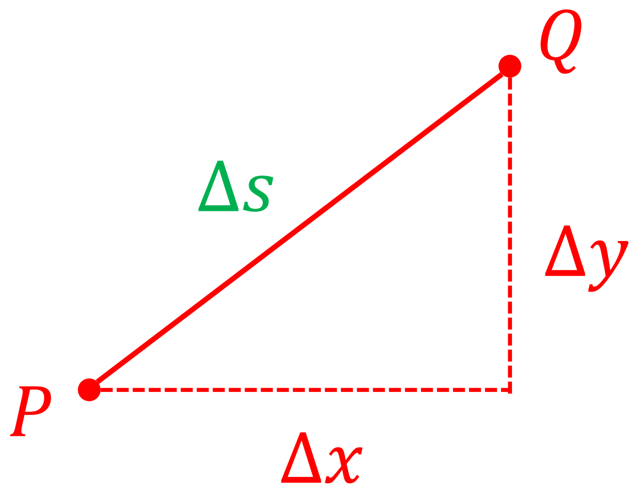


Figure 11

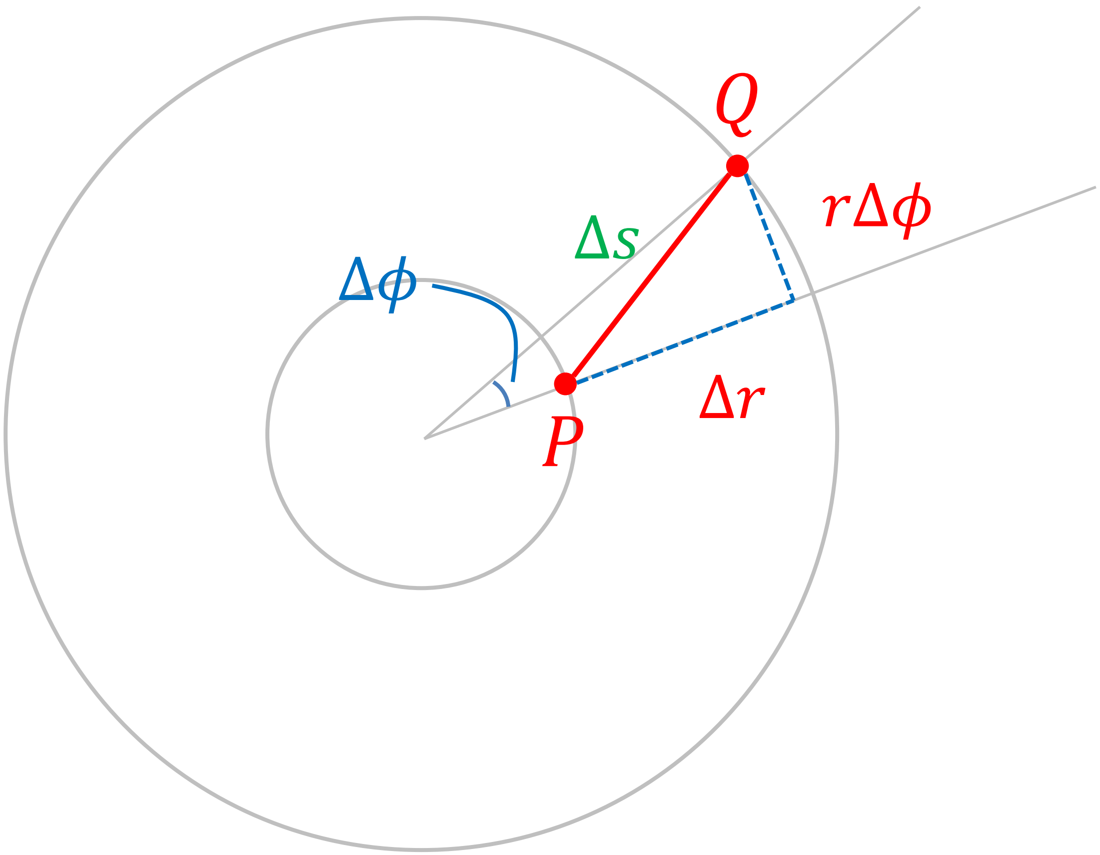


Figure 12

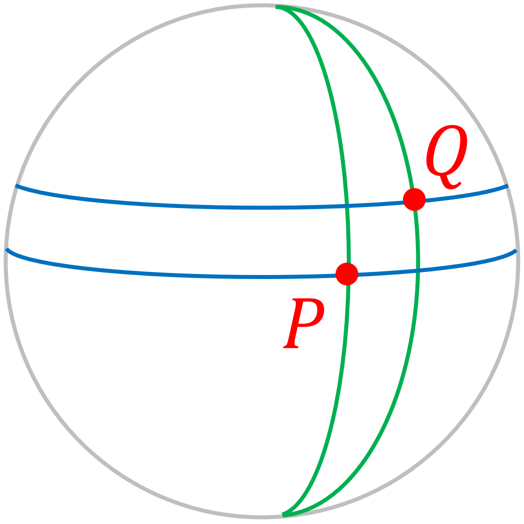


Figure 13

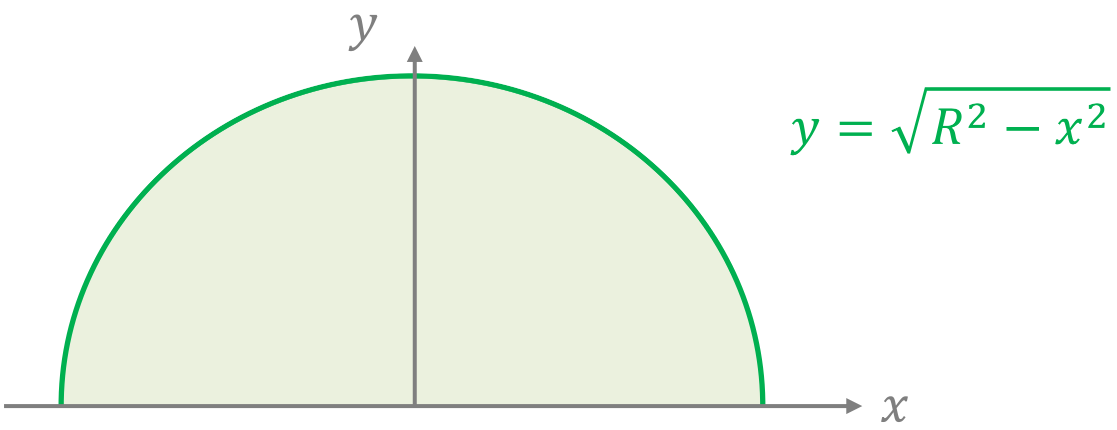


Figure 14a

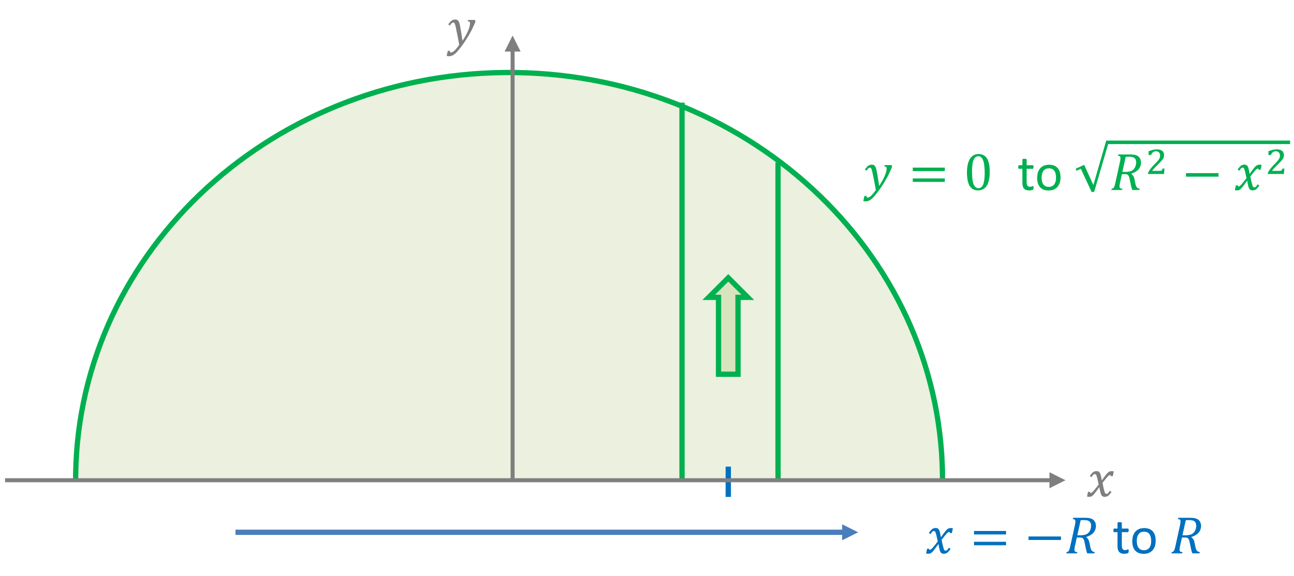


Figure 14b

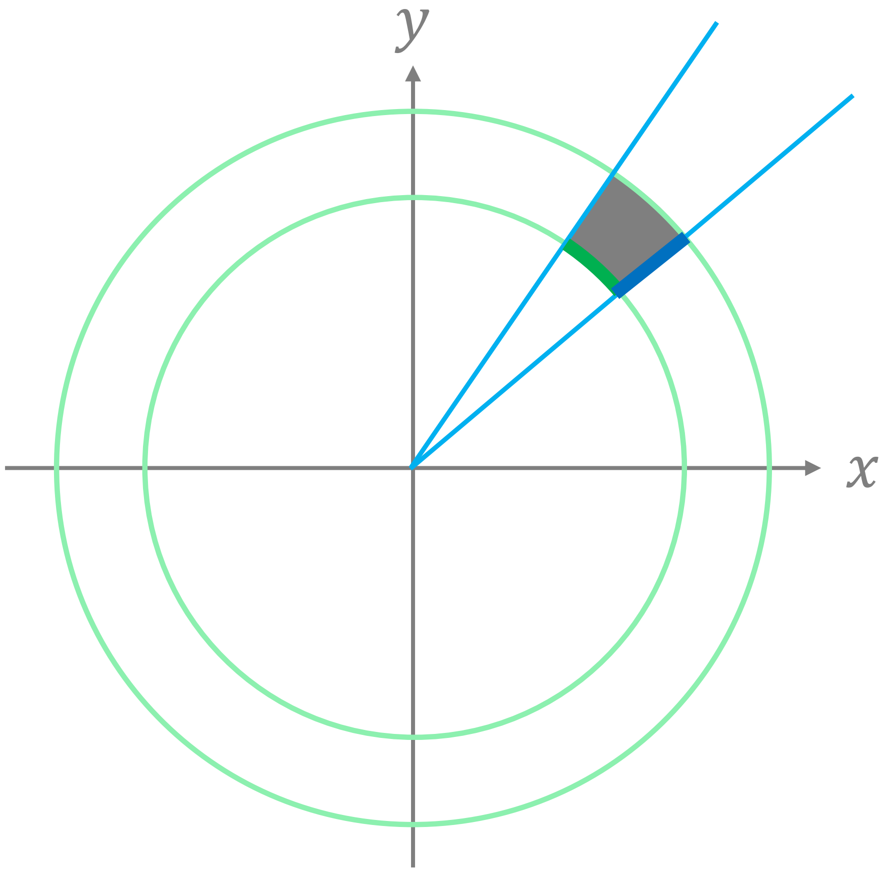


Figure 15

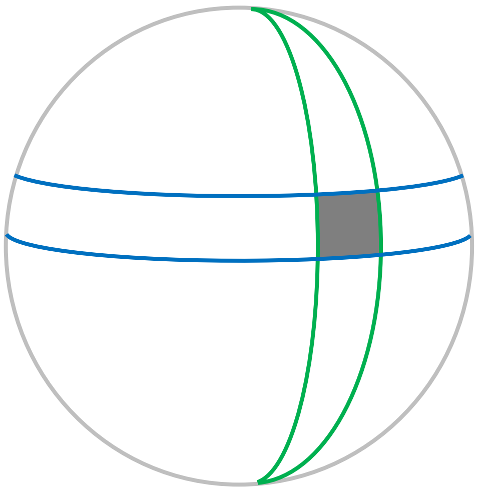


Figure 16a

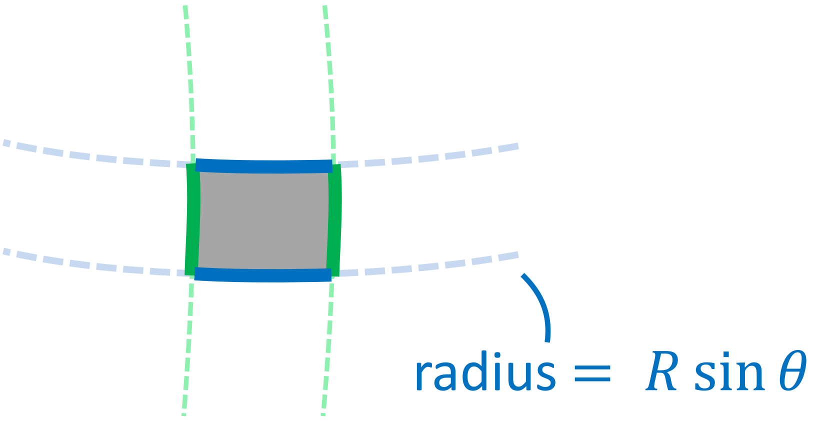


Figure 16b