Complex variables: Part I

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This module introduces complex numbers and their arithmetic and algebra. Two advanced topics, analytic functions and Cauchy integrals, are sketched in the next module.

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1 Motivation

Physics is about the relationship between measurable quantities, e.g., force and acceleration related as $F = ma$. Measurable quantities are real. So why should we care about complex numbers?

- Some formulas become simpler if we consider their complex generalizations. For example, the formula (see the module on Elementary Functions)

$$e^{i\theta} \;=\; \cos\theta + i\sin\theta \qquad (1)$$

makes the addition formula for trigonometric functions easier to understand. You probably know that simple harmonic motion is the projection onto one axis (say the x-axis) of uniform circular motion. This statement is just an application of (1), with $\theta = \omega t$ and ω being the angular frequency of circular motion.

- Related to the above, a function such as $\exp(i\omega t)$ has a simple property under differentiation:

$$\begin{aligned}(d/dt)\, e^{i\omega t} &= i\omega\, e^{i\omega t} \\ (d/dt) &\mapsto i\omega\end{aligned} \qquad (2)$$

turning differentiation into multiplication. For this reason, we often express oscillatory functions such as $\cos\omega t$ as the real part of complex exponentials.

- For a smooth function $f(x)$, we can learn a lot more by generalizing it to complex arguments $f(z)$. If $f(z)$ is known on a closed curve in the complex plane, it would be completely known everywhere inside the curve — a very strong property that has no counterpart for real arguments. Functional dependences in physics are usually smooth enough for this condition to apply. For example, this concept allows us to predict the dispersion of an optical medium (how the refractive index changes with wavelength) from a knowledge of its absorption spectrum. Also, many special functions encountered in physics (for which real arguments are involved) become better understood if we extend to complex arguments.

- Finally, in quantum mechanics, the central object is a complex wavefunction $\psi(x,t)$, whose absolute square $|\psi(x,t)|^2$ is the measurable probability density. In this case, there is no way to avoid complex numbers — the imaginary part is not just tagged on for calculational convenience.

2 Arithmetic

2.1 Complex numbers

Square root of minus one
There is no real number x such that $x^2 = -1$. We extend the real numbers by defining a new number i such that

$$i^2 = -1 \tag{3}$$

Of course, there is another solution $-i$:

$$(-i)^2 = -1 \tag{4}$$

By the way, electrical engineers tend to use j to denote $\sqrt{-1}$, reserving i to denote the current.

Complex numbers
Complex numbers are numbers of the form

$$z = x + yi \tag{5}$$

where x, y are real numbers, respectively called the real and imaginary parts:

$$\begin{aligned} \Re z &= x \\ \Im z &= y \end{aligned} \tag{6}$$

Complex numbers can be represented on the *complex plane*, which is the 2D plane of (x, y). (See below on vector representation.)

Complex conjugate
Given a complex number z, define the *complex conjugate* or simply *conjugate*, denoted as z^*, as follows:

$$\begin{aligned} z &= x + yi \\ z^* &= x - yi \end{aligned} \tag{7}$$

Note that

$$z^* z = x^2 + y^2 \tag{8}$$

is real and non-negative.

Notation: Physicists tend to denote the conjugate as z^* whereas mathematicians tend to denote it as $\bar{z}$.

2.2 Addition, subtraction and multiplication by a real number

If $z_j = x_j + y_j i$, and $z = z_1 + z_2$, then

$$\begin{aligned} z &= (x_1 + y_1 i) + (x_2 + y_2 i) \\ &= (x_1 + x_2) + (y_1 + y_2) i \\ x &= \Re z = x_1 + x_2 \\ y &= \Im z = y_1 + y_2 \end{aligned} \tag{9}$$

In other words the real and imaginary parts add independently. The same property holds for subtraction.

Multiplication by a real number β also works separately on the real part and the imaginary part:

$$\beta z = (\beta x) + (\beta y) i \tag{10}$$

2.3 Vector representation

A complex number z can be represented as a point $P = (x, y)$ on a 2D plane (**Figure 1**), or as the vector

$$\vec{z} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} \tag{11}$$

where $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ are unit vectors in the x and y directions. Addition, subtraction and multiplication by a real number all work exactly as for the analogous vectors.

2.4 Multiplication and division

But complex numbers have a much richer structure than mere vectors: multiplication and division are also possible.

Here multiplication means a commutative product ($z_1 \cdot z_2 = z_2 \cdot z_1$) with the result being another complex number. The analogue for vectors does not exist. The analogue of division certainly does not exist.

Multiplication
Let $z_j = x_j + y_j i$ and $z = z_1 z_2$:

$$z = (x_1 + y_1 i) \cdot (x_2 + y_2 i)$$

$$
\begin{aligned}
&= (x_1x_2 - y_1y_2) + (x_1y_2 + y_1x_2)i \\
x &= \Re z = x_1x_2 - y_1y_2 \\
y &= \Im z = x_1y_2 + y_1x_2 \qquad (12)
\end{aligned}
$$

Problem 1
Find $(3+4i)\cdot(5+6i)$. §

Division
Division is a bit trickier. Let us start with an example.

Example 1
Write $1/(3+4i)$ in the standard form $x+yi$.
To do so, we multiply and divide by the complex conjugate:

$$
\begin{aligned}
\frac{1}{3+4i} &= \frac{1}{3+4i}\cdot\frac{3-4i}{3-4i} \\
&= \frac{3-4i}{25} = (3/25) - (4/25)i \qquad (13)
\end{aligned}
$$

The denominator z^*z is always real (and positive); then the only i occurs in the numerator and we get the standard form. §

Thus in general we have, provided that x and y are not both zero,

$$
\frac{1}{x+yi} = \frac{x-yi}{x^2+y^2} \qquad (14)
$$

Problem 3
Find $(5+12i)/(3+4i)$ in standard form. §

To summarize, we can perform all arithmetic operations (add, subtract, multiply, divide — in the last case provided the denominator is not zero) with complex numbers, and the answer is still a complex number. Mathematicians say that the complex numbers form a *field*.

Complex arithmetic has all the properties of 2D real vectors, and *much more besides*.

3 Polar representation

3.1 Modulus and argument

If we regard a complex number $z = x+yi$ as a point (x,y) on a 2D plane, then there is a natural polar representation in terms of a length and an angle:

$$
\begin{aligned}
|z| &= \sqrt{x^2+y^2} \\
\arg z = \theta &= \arctan y/x \qquad (15)
\end{aligned}
$$

The length is called the *modulus* and the angle is called the *argument* or *phase*. The arctan is only a shorthand: in reality we need to look at the individual signs of x and y. For example, the points $P = (x,y) = (3,4)$ is in the first quadrant, while $Q = (x',y') = (-3,-4)$ is in the third quadrant, even though the ratios y/x and y'/x' and hence the arctan values are the same.

The reverse transformation is

$$
\begin{aligned}
x &= |z|\cos\theta \\
y &= |z|\sin\theta \qquad (16)
\end{aligned}
$$

But using the Euler formula (see the module on elementary functions)

$$
e^{i\theta} = \cos\theta + i\sin\theta \qquad (17)
$$

we see that

$$
\boxed{z = |z|\,e^{i\theta}} \qquad (18)
$$

An amazing identity
By putting $\theta = \pi$ into (18), we get

$$
\boxed{e^{i\pi} = -1} \qquad (19)
$$

an amazing formula that relates three important numbers in mathematics: e, π and i.

3.2 Multiplication and division

Multiplication becomes easy in polar representation. If

$$
z_j = |z_j|\,e^{i\theta_j} \qquad (20)
$$

then the product $z = z_1\cdot z_2$ has the polar representation (18), with

$$
\begin{aligned}
|z| &= |z_1|\cdot|z_2| \\
\theta &= \theta_1 + \theta_2 \qquad (21)
\end{aligned}
$$

with the obvious changes for division.

Problem 4
Use the polar representation for $5+12i$ and $3+4i$ (see Problem 3) to calculate $(5+12i)/(3+4i)$ in polar coordinates. Then express as $x+yi$ and compare with Problem 2. §

3.3 Roots

It is now easy to find one square root of any complex number z given by (18). Evidently it is

$$\sqrt{z} \quad = \quad |z|^{1/2}\, e^{i\theta/2} \tag{22}$$

and more generally the nth root is

$$z^{1/n} \quad = \quad |z|^{1/n}\, e^{i\theta/n} \tag{23}$$

In these formulas, the first factor is the conventional real root of a positive real number.

But a square root should have two possibilities and the nth root should have n possibilities. The solutions are

$$\begin{aligned} \sqrt{z} &= \pm|z|^{1/2}\, e^{i\theta/2} \\ z^{1/n} &= |z|^{1/n}\, e^{i\theta/n}\, e^{2(k/n)\pi i} \end{aligned} \tag{24}$$

for $k = 0, \ldots, (n-1)$. We note that (a) the last factor in the second formula is any one of the nth roots of unity, i.e.,

$$\left[e^{2(k/n)\pi i}\right]^n \quad = \quad e^{2k\pi i} = 1 \tag{25}$$

and (b) the first equation in (24) is a special case for $n = 2$.

Problem 5
Find the cube roots of 1 and of -1 and express these both in polar representation and in the form $x + yi$. Check by taking the cube of $x + yi$. §

4 Algebra

4.1 Quadratic equations

Using complex numbers, the equation

$$z^2 = \Delta \tag{26}$$

has a solution (indeed two solutions) for any Δ, positive or negative (even complex). More generally, the quadratic equation

$$az^2 + bz + c \quad = \quad 0 \tag{27}$$

has two solutions

$$\begin{aligned} z &= \frac{-b \pm \sqrt{\Delta}}{2a} \\ \Delta &= b^2 - 4ac \end{aligned} \tag{28}$$

You should know the proof by completing squares, and the crucial step is taking the square root of Δ just as in (26). In most applications in physics, the coefficients a, b, c are real, but all the above is correct even for complex coefficients.

4.2 Counting roots

The quadratic in general has two roots (for $\Delta > 0$ or $\Delta < 0$). But in the special case $\Delta = 0$, the $+$ or $-$ sign gives the same root — it appears if there is only one root.

A better convention is to count this as two roots. In general, for an algebraic equation

$$f(z) \quad = \quad 0 \tag{29}$$

A root ζ is a value of z for which (29) is satisfied. But if we have

$$f(\zeta) = f'(\zeta) = \ldots = f^{(n-1)}(\zeta) \quad = \quad 0 \tag{30}$$

with the next derivative nonzero, in other words if the Taylor series about ζ starts with

$$f(z) \quad = \quad a(z - \zeta)^n + \ldots \tag{31}$$

then we say the root is of order n, and count it as n roots.

Problem 6
Consider (27) for the case $\Delta = 0$. Show that the root has order 2 in the sense defined above. §

Problem 7
Consider the quadratic $z^2 + 2z + c = 0$ and plot the locus of the two roots as c varies continuously from 0 to 2. This example illustrates a general property: The solutions change continuously with the change of parameters; they may "collide" but never "disappear". §

4.3 General polynomials

Consider a general polynomial equation of order n:

$$\sum_{k=n}^{0} a_k z^k \quad = \quad 0 \tag{32}$$

where $a_n \neq 0$. There are exactly n roots, counting multiple roots if any in the sense defined above. The proof requires some tools of complex analysis, and will be given in the next module.

The polynomials that appear in physics usually have real coefficients a_k. In this case, if ζ is a root, then so is ζ^* (which would be a different root if and only if $\Im\,\zeta \neq 0$).

4.4 The square root function*

**This subsection is more advanced and can be skipped.*

Multiple roots
We have already learnt how to take the square root, but there is some subtlety when we regard the square root as a *function*:

$$f(z) \quad = \quad \sqrt{z} \qquad (33)$$

The reason is this: There are two solutions, and we have to make a consistent choice so that $f(z)$ is continuous. Let $\arg z = \theta$, by convention defined to be

$$0 \leq \theta < 2\pi \qquad (34)$$

Then we can write z in either of the two ways

$$z \quad = \quad |z|\, e^{i\theta} \quad \text{or} \quad |z|\, e^{i(\theta+2\pi)} \qquad (35)$$

Then using the rule that the argument is to be halved, we have

$$\sqrt{z} \quad = \quad |z|^{1/2}\, e^{i\theta/2} \quad \text{or} \quad |z|^{1/2}\, e^{i(\theta/2+\pi)} \qquad (36)$$

The extra phase $i\pi$ in the second solution is just a minus sign. This explanation gives us another perspective: multiple roots appear because we can assign different phases to z (differing by 2π) — a complication that was already present before we take the square root.

Consider an example: take z anticlockwise on the unit circle ($|z| = 1$), starting from the positive real axis, going through the points A, B, C, D, E in **Figure 2**. We take the first root in (35) and (36). Denote $\theta = \arg z$, $\theta' = \arg \sqrt{z}$.

	θ	θ'
A	0^+	0^+
B	$\pi/2$	$\pi/4$
C	π	$\pi/2$
D	$3\pi/2$	$3\pi/4$
A'	2π	π

Table 1. The arguments $\theta = \arg z$ and $\theta' = \arg \sqrt{z}$, for z going around the unit circle. The complex plane is cut along the positive real axis, as in Figure 3. The symbol 0^+ means an infinitesimal positive number.

Problem 8
Write $\sqrt{z}$ in the standard form $x + yi$ for each of the points A, B, C, D, A' in Table 1. §

Need for a cut
We now have a problem: the points A' and A are almost the same; yet $\sqrt{z}$ is quite different: it is $+1$ at A and -1 at A'. If we allow these two points to be regarded as "nearby", the function would be discontinuous. There are two ways out. (a) When θ goes beyond 2π, we imagine z going onto a *second sheet*, i.e., another copy of the complex plane. We shall not deal with this point of view here. (b) We simply do not allow A' and A to be connected, by declaring the positive real axis to be a barrier that cannot be crossed. In other words, we *cut* the plane along the positive real axis, as indicated in **Figure 2**. The cut is simply a way of saying that we assign the value of θ as in Table 1 (and not, for example, 2π larger).

But we can cut the complex plane in other ways, for example, as in **Figure 3**. If that is done, the arguments would be as shown in Table 2. There is now a discontinuity in $\sqrt{z}$ between the point C (just above the negative real axis) and the point C' (just below the negative real axis), on different sides of the cut. Now there is no discontinuity between A' and A.

	θ	θ'
A	0^+	0^+
B	$\pi/2$	$\pi/4$
C	π	$\pi/2$
C'	$-\pi$	$-\pi/2$
D	$-\pi/2$	$-\pi/4$
A'	0^-	0^-

Table 2. The arguments $\theta = \arg z$ and $\theta' = \arg \sqrt{z}$ for z going around the unit circle. The complex plane is cut along the negative real axis, as shown in Figure 4. The symbol 0^+ (0^-) means an infinitesimal positive (negative) number.

Problem 9
Write $\sqrt{z}$ in the standard form $x + yi$ for each of the points A, B, C, C', D, A' in Table 2. Compare with the previous problem. §

4.5 The logarithm function*

**This subsection is more advanced and can be skipped.*

In elementary discussions of the logarithm, we learnt about the function $\ln x$, but only for x real and positive.[1] What about $\ln z$ for other values: negative real numbers or in general complex numbers?

$$\begin{aligned} \ln z &= \ln\left(|z|\, e^{i\theta}\right) \\ &= \ln|z| + i\theta \end{aligned} \qquad (37)$$

where the first term is the log of a positive real number, and the second term is obtained by the usual rule of picking out the exponent. Thus, if z is not a positive real number, its logarithm has an imaginary part.

Example 2
Find $\ln(-4)$.
Write

$$-4 = 4\, e^{i\pi} \qquad (38)$$

Then

$$\begin{aligned} \ln(-4) &= \ln 4 + i\pi \\ &= 1.39 + 3.14i \end{aligned} \qquad (39)$$

But note: If we write the argument as $-\pi$, we would get a different answer. §

Thus there is the same problem as with $\sqrt{z}$: what do we take the value of θ to be, and is it continuous? Exactly as before, we need (a) a convention to define θ, and (b) a cut to prevent discontinuities. The cut can be placed on the positive real axis (and a positive real number is regarded as just above the cut), or it can be placed on the negative real axis.

Problem 10
Let z_j, $j = 1, 2, 3$, be the three cube roots of unity. (See Problem 4). Find $\ln z$ using the cut prescriptions in Figure 3 and Figure 4. Thus there should be 6 answers. §

[1] For any other base b, just divide by $\ln b$.

$$z = x + y\,i$$ complex number

$$\vec{z} = x\hat{\imath} + y\hat{\jmath}$$ vector

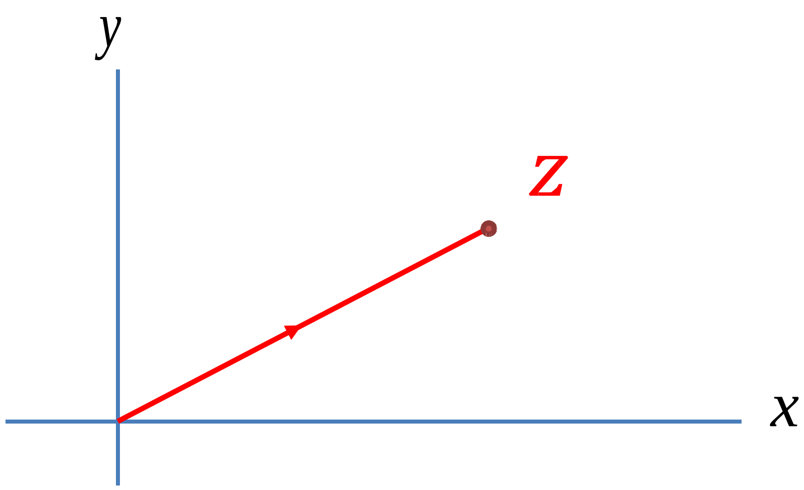

Figure 1: A complex number can also be regarded as a 2D vector

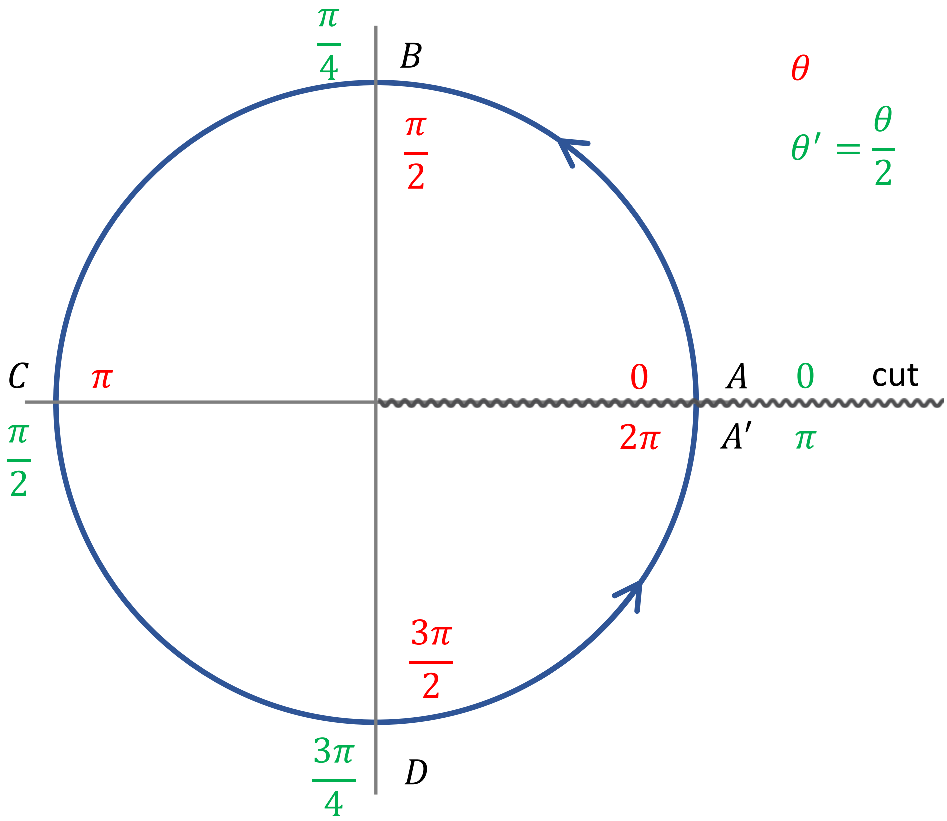

Figure 2: Choice of phase for the square root function; cut on positive real axis

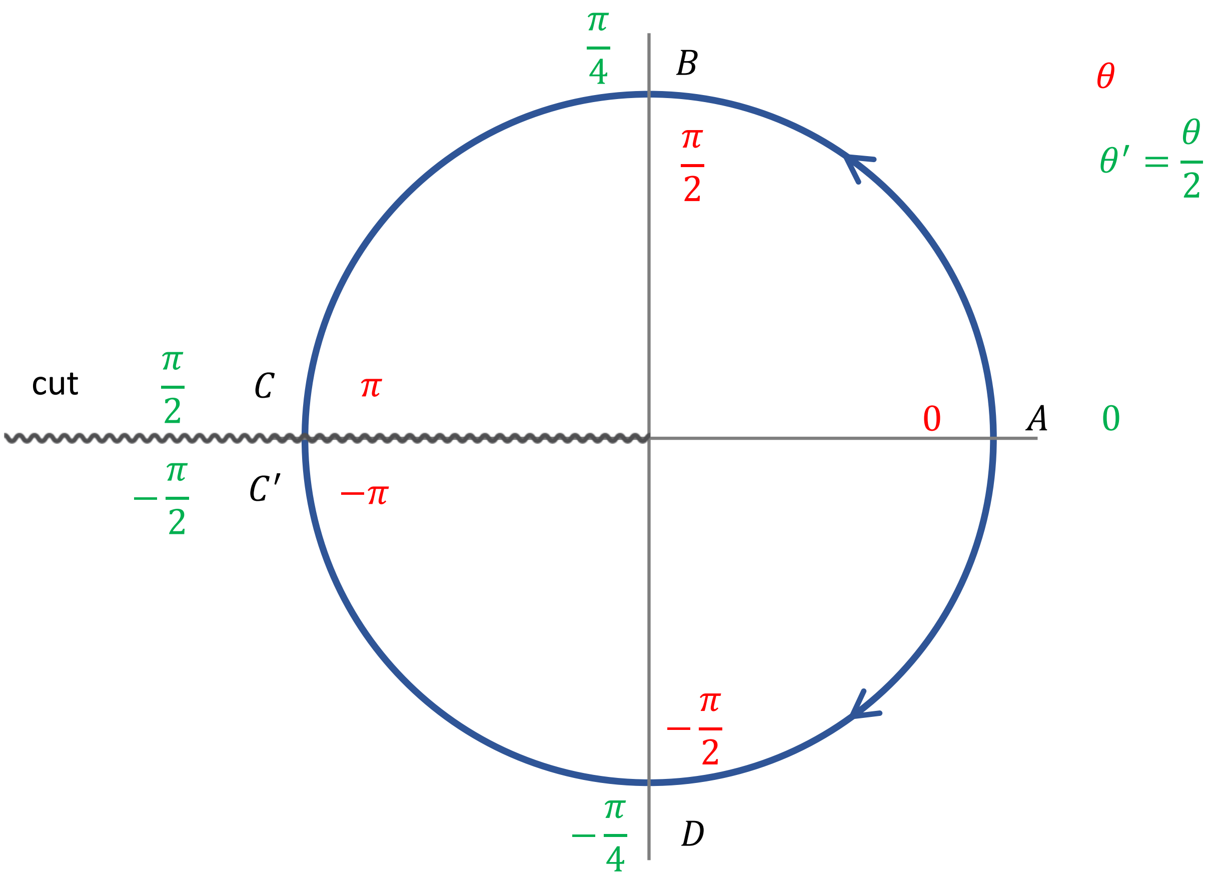

Figure 3: Choice of phase for the square root function; cut on negative real axis