Power series

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This module introduces power series, including the Taylor series for any given function, together with some applications.

Contents

1 Power series

This module deals with function such as $f(x) = \sin x$. The argument x must be dimensionless (e.g., an angle is measured in radians, which carries no dimension), and we assume the function is sufficiently smooth.[1]

Such a function can (usually) be expressed as an infinite power series valid for a certain range of x that is "not too large".

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

1.1 Range of convergence and example

How large is "not too large"? Instead of any formal theory, we illustrate by examples: in the first example below, a_n do not decrease, so x cannot be large; in the example in Section 2, a_n decreases rapidly, and the series converges for all x.

Geometric series

The series defined by $a_n = 1$ converges provided $|x| < 1$, in fact with the result

$$S = \sum_{n=0}^{\infty} x^n = (1 - x)^{-1} \tag{2}$$

The proof is outlined below in Problem 1, but the idea is already familiar to young students who have learnt recurring decimals. Take $x = 0.1$, write out the series as a recurring decimal, and note that multiplication by 0.1 is equivalent to shifting by one digit:

$$\begin{aligned} S &= 1.111111\dots \\ (0.1)S &= 0.111111\dots \\ (0.9)S &= 1 \\ S &= \frac{1}{0.9} = \frac{10}{9} \end{aligned}$$

[1] We leave it for mathematicians to worry about what "sufficiently" means.

Problem 1
Consider the partial sum

$$S_N = \sum_{n=0}^{N} x^n \tag{3}$$

and compare with xS_N. (Hint: shift one term.) Hence show that

$$(1-x)S_N = 1 - x^{N+1} \tag{4}$$

By taking $N \to \infty$, prove (2) for $|x| < 1$. §

1.2 Binomial series

This subsection generalizes the familiar formulas such as

$$\begin{aligned} (1+x)^2 &= 1 + 2x + x^2 \\ (1+x)^3 &= 1 + 3x + 3x^2 + x^3 \\ (1+x)^4 &= 1 + 4x + 6x^2 + 4x^3 + x^4 \end{aligned} \tag{5}$$

Positive integer powers
For a positive integer n, we know

$$\begin{aligned} & (1+x)^n \\ = & \ 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots \\ = & \sum_{k=0} \frac{n(n-1)\ldots(n-k+1)}{k!}\, x^k \end{aligned} \tag{6}$$

Deliberately, we have not indicated an upper limit to the sum; in principle, allow k to go to infinity. But provided n is an integer, for $k > n$ the numerator in the coefficient will encounter a factor of zero. Thus, this is a finite series, and therefore valid for any x.

Other powers
It turns out that the same formula works (with two qualifications, as below) even if the power is not an integer:

$$\begin{aligned} & (1+x)^a \\ = & \ 1 + ax + \frac{a(a-1)}{2}x^2 + \ldots \\ = & \sum_{k=0} \frac{a(a-1)\ldots(a-k+1)}{k!}\, x^k \end{aligned} \tag{7}$$

There are two differences. (a) If a is not a positive integer, the coefficient is never zero, and the series is an infinite series. (b) For an infinite series, one has to worry about convergence. This series converges if (and only if) $|x| < 1$. A formal proof of (7) is given later.

Problem 2
Take $a = -1$ and explicitly work out the coefficients in (7). Compare with (2). §

Problem 3
Take $a = 1/2$ and and write out three terms of the expansion. §

1.3 Using a power series

Given a series such as (1), there are three ways in which it can be used.

- Take *all* the terms and use the sum to obtain the *exact* answer. This is what mathematicians tend to do. Our proof of (2) belongs to this category.

- Take *many* terms and use the sum to obtain an *accurate* answer. This is what computer scientists tend to do. For example, this is one main method used for evaluating special functions. See Problems 5, 6.

- For some x that is *much* smaller than unity, we can take *a few* terms and get a pretty good answer. This is what physicists often do. See Examples 1, 2 and Problem 7.

These uses are illustrated by some examples.

Problem 4
Imagine using the series in (2) to evaluate $(1 - 0.99)^{-1}$. How many terms do you need to get an absolute accuracy of 0.001? §

Problem 5
Consider the function $\exp x$ introduced in (17) in the next Section. Evaluate $\exp 0.2$ to an accuracy of 0.001. Use only the arithmetic functions (adding, subtracting, multiplying and dividing) in your calculator. §

Problem 6
The Bessel function is defined by the series

$$J_0(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2/4)^n}{(n!)^2} \tag{8}$$

(a) Write out the first four terms explicitly.
(b) Evaluate $J_0(0.3)$ to 3 significant figures. §

Example 1
Consider the following method to calculate $\sqrt{2}$.

$$\begin{aligned} & \sqrt{2} \\ = & \sqrt{\frac{200}{100}} = (1/10) \cdot \sqrt{200} \\ = & (1/10)\ (196+4)^{1/2} \\ = & (1/10)\sqrt{196}\ (1+4/196)^{1/2} \end{aligned} \tag{9}$$

Note that $196 = 14^2$ and use the series

$$(1+x)^{1/2} = 1 + (1/2)x + \ldots \tag{10}$$

applied to $x = 4/196 \ll 1$, so that we can drop the x^2 term etc.:

$$(1+4/196)^{1/2} \approx 1 + 2/196 \approx 1.01 \tag{11}$$

Thus

$$\sqrt{2} \approx (1/10) \times 14 \times 1.01 = 1.414 \tag{12}$$

Four figures are obtained without even a calculator. Try using

$$2,000,000 = 1414^2 + 604 \tag{13}$$

for an even more accurate result.

Problem 7
Find $\sqrt{3}$ in a similar way. Hint: 17^2 is just a little bit less than 300. §

Example 2
In relativity, the energy E of a particle of mass m moving at velocity v is given by

$$E = \frac{mc^2}{\sqrt{1-v^2/c^2}} \tag{14}$$

where c is the velocity of light. For many cases, v/c is very small, so $x \equiv v^2/c^2$ is tiny, and its second and higher powers can be ignored. (For example, for a plane flying at approximately the speed of sound, $v \sim 300$ m s^{-1}, $v/c \sim 10^{-6}$, $x \equiv (v/c)^2 \sim 10^{-12}$.) Thus

$$\begin{aligned} & \left[1-(v/c)^2\right]^{-1/2} \\ \approx & 1 + \left[-\frac{1}{2}\right]\left[-(v/c)^2\right] \\ = & 1 + \frac{1}{2}\frac{v^2}{c^2} \end{aligned} \tag{15}$$

Thus

$$E \approx mc^2 + \frac{1}{2}mv^2 \tag{16}$$

The first term is the energy when the particle is not moving (the rest energy), and the second term is the additional energy due to motion (kinetic energy). Thus the Newtonian form of the kientic energy is recovered. §

Problem 8
A ball of 1 kg is moving at a speed of 3 m s^{-1}. Find in units of J (a) its rest energy; (b) its Newtonian kinetic energy; (c) the *error* in its Newtonian kinetic energy. §

Very often, keeping just one non-trivial term in a power series expansion is a good way of obtaining a simpler yet accurate expression — and to connect a general theory or expression with a more restricted one that applies in a limited domain.

2 Exponential function

2.1 Definition

The series defined by $a_n = 1/n!$ converges for any value of x. Define the resulting function as

$$\exp x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{17}$$

2.2 Multiplicative property

The exponential function has the important property that

$$\exp x \cdot \exp y = \exp(x+y) \tag{18}$$

To establish this property, we have to check that

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \cdot \left(\sum_{m=0}^{\infty} \frac{y^m}{m!}\right) \stackrel{?}{=} \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!} \tag{19}$$

Check a few terms
In case you are not familiar with manipulating infinite series, let us proceed step by step. Look at just the first few terms:

$$\begin{aligned} & \left(1+x+x^2/2!+\ldots\right) \cdot \left(1+y+y^2/2!+\ldots\right) \\ \stackrel{?}{=} & 1 + (x+y) + (x+y)^2/2! + \ldots \end{aligned} \tag{20}$$

Arrange terms by the total power k of x and y. (To make this more systematic, we can introduce a formal parameter μ: replace $x \mapsto \mu x$, $y \mapsto \mu y$ and count powers of μ.) The $k = 0$ and $k = 1$ terms are obvious, and the $k = 2$ terms are

$$\begin{aligned} \text{LHS} &= x^2/2! + xy + y^2/2! \\ \text{RHS} &= (x+y)^2/2! \end{aligned} \tag{21}$$

which are clearly equal.

Problem 9
Carry out the same verification for $k = 3$ and $k = 4$. §

Formal proof
Look at terms with total power k:

$$\begin{aligned} & \text{LHS} \\ = & \sum_{m+n=k} \frac{x^n}{n!} \cdot \frac{y^m}{m!} \\ = & \sum_{n=0}^{k} \frac{x^n y^{k-n}}{n!(k-n)!} \\ = & \frac{1}{k!} \sum_{n=0}^{k} \frac{k!}{n!(k-n)!} x^n y^{k-n} \\ = & \frac{1}{k!}(x+y)^k = \text{RHS} \end{aligned} \tag{22}$$

2.3 The number e

Define the number e as

$$\begin{aligned} e &\equiv \exp 1 \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots \\ &= 2.71828\ldots \end{aligned} \tag{23}$$

Thus we have, for example,

$$\begin{aligned} & e^2 = e \cdot e \\ = & \exp 1 \cdot \exp 1 = \exp 2 \end{aligned} \tag{24}$$

More generally,

$$e^x = \exp x \tag{25}$$

for any x. Thus the exponential function is just a power. Henceforth we shall interchangeably write e^x or $\exp x$.

2.4 Some properties

Here we list only a few elementary properties of the exponential function. Other properties, especially about (a) its derivative and (b) its relationship with trigonometric functions, will be discussed later.

Monotonic
The function $\exp x$ is monotonically increasing: if $x < y$, then $\exp x < \exp y$. To see this, simply compare similar terms in the two expansions. (This proof works only for $x, y > 0$. How would you generalize the proof to other cases?)

Never zero
The function $\exp x$ has no zeros. To see this, we only have to note that there is a reciprocal:

$$\exp x \cdot \exp(-x) = 1 \tag{26}$$

Grows faster than any power
An exponential grows faster than any power, in the following sense. For any $a, b > 0$ and fixed $n \geq 0$, for sufficiently large $x > 0$ we have

$$e^{ax} > bx^n \tag{27}$$

To prove this, we just pick one term in the series on the LHS (the omitted terms being all positive), and we only need to prove

$$\frac{1}{(n+1)!} a^{n+1} x^{n+1} > bx^n \tag{28}$$

which will hold provided

$$x > x_0 \equiv (n+1)! \, \frac{b}{a^{n+1}} \tag{29}$$

A simple corollary is that for any $a, b > 0$ and fixed $n \geq 0$, then for sufficiently large x we have

$$e^{-ax} < bx^{-n} \tag{30}$$

for sufficiently large x.

3 Differentiation and integration

3.1 Term by term

Provided the power series converges "sufficiently" well,[2] then we can differentiate or integrate term

[2] Technically, that the convergence is uniform.

by term, e.g., given

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{31}$$

then

$$\begin{aligned} \frac{d}{dx} f(x) &= \sum_{n=0}^{\infty} n a_n x^{n-1} \\ \int f(x)\, dx &= c_0 + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \end{aligned} \tag{32}$$

In the derivative we can omit the $n = 0$ term; in the integral there is an unknown constant of integration c_0.

3.2 Examples

Problem 10
Differentiate (2) and hence obtain power series expansions for

$$(1-x)^{-2} \qquad , \qquad (1-x)^{-3} \tag{33}$$

Also check (at least the first of these) by multiplying (2) by itself and sorting out like powers. §

The next two problems assume that you know about the natural logarithm.

Problem 11
Integrate (2) and obtain a power series for $\ln(1-x)$. Note that $\int x^{-1}\, dx = \ln x$. Also obtain the series for $\ln(1+x)$. §

Problem 12
There is a simple rule in finance: If an investment increases in value by p percent per year (and p is not too large), then it will take $n = 70/p$ years for it to double. Prove this by starting with

$$(1 + p/100)^n = 2 \tag{34}$$

Take the natural log and expand in the small parameter $x = p/100$. Explain why it is not appropriate to expand (34) directly. §

Problem 13
Take the series for $\exp x$ and show that

$$\frac{d}{dx} e^x = e^x \tag{35}$$

i.e., the exponential function is its own derivative. What is the derivative of e^{ax}? §

Problem 14
From (2) we have (expanding in $y = -x^2$)

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots \tag{36}$$

valid for all $|x| < 1$. Integrate term by term and hence prove the following series for the arc tangent:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots \tag{37}$$

Apply this to $x = 1$ (which is "barely" OK, in the sense that the resulting series is conditionally convergent), to obtain the following series for evaluating π:

$$\begin{aligned} \frac{\pi}{4} &= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \\ &= 2\left(\frac{1}{1\cdot 3} + \frac{1}{5\cdot 7} + \ldots\right) \end{aligned} \tag{38}$$

The latter form converges much better. Evaluate π to 3 significant figures. §

4 Taylor series

4.1 Finding the coefficients

Given a function $f(x)$, how do we find the coefficients a_n in the power series (1)?
(0) Start with the expansion of f and evaluate at $x = 0$:

$$f(0) = a_0 \tag{39}$$

(1) Next differentiate the expansion once:

$$f'(x) = a_1 + 2a_2\, x + 3a_2\, x^2 + \ldots \tag{40}$$

Now evaluate this at $x = 0$:

$$f'(0) = a_1 \tag{41}$$

(2) Differentiate one more time:

$$f^{(2)}(x) = 2a_2 + 3\cdot 2a_2\, x + 4\cdot 3\, x^2 + \ldots \tag{42}$$

Now evaluate this at $x = 0$:

$$f^{(2)}(0) = 2a_2 \tag{43}$$

Here we adopt the notation that $f^{(n)}$ means the nth derivative.

It is easy to see that after doing this n times

$$f^{(n)}(0) = n!\, a_n \tag{44}$$

Turning this around

$$\boxed{a_n = f^{(n)}(0)/n!} \tag{45}$$

or, putting this back into (1),

$$f(x) = \sum_{n=0}^{\infty} [f^{(n)}(0)/n!]\, x^n \tag{46}$$

This is the the Taylor series for expanding around $x = 0$.

4.2 Expanding around another point

Consider the function

$$g(h) = f(x+h) \tag{47}$$

By applying the above result

$$g(h) = \sum_{n=0}^{\infty} [g^{(n)}(0)/n!]\, h^n \tag{48}$$

But

$$g^{(n)}(0) = f^{(n)}(x) \tag{49}$$

so we have

$$f(x+h) = \sum_{n=0}^{\infty} [f^{(n)}(x)/n!]\, h^n \tag{50}$$

which is a more general form useful for expanding around x rather than 0. Strictly speaking the special case of (46) is called a Maclaurin series, whereas the general form (50) is called a Taylor series.

4.3 Some examples

Example 3
Let $f(x) = (1+x)^a$. Then

$$\begin{aligned} f(x) &= (1+x)^a \\ f'(x) &= a\,(1+x)^{a-1} \\ f^{(2)}(x) &= a(a-1)\,(1+x)^{a-2} \\ f^{(3)}(x) &= a(a-1)(a-2)\,(1+x)^{a-3} \end{aligned} \tag{51}$$

etc., so evaluating at $x = 0$,

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= a \\ f^{(2)}(0) &= a(a-1) \\ f^{(3)}(0) &= a(a-1)(a-2) \end{aligned} \tag{52}$$

and hence we get

$$\begin{aligned} &(1+x)^a \\ &= 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^3 \\ &\quad + \ldots \end{aligned} \tag{53}$$

valid for $|x| < 1$.

We make several remarks. (a) The above is valid for any power a, which does not have to be an integer. (b) If a is a positive integer, the series terminates, and the result is a polynomial — the familiar binomial expansion. §

Problem 15
Take $a = -1$ and change $x \mapsto -x$. Hence derive (2). §

5 Newton's method

5.1 Formulation

In physics we often have to solve algebraic equations numerically. Such equations in one variable can always be written as

$$f(x) = 0 \tag{54}$$

for some smooth function f.

Suppose we have an initial guess x_0 that is not too far off. Thus we consider only those $x = x_0 + \xi$ where ξ is small. Then

$$0 = f(x_0 + \xi) \approx f(x_0) + \xi f'(x_0) \tag{55}$$

Based on this approximation, the zero should occur for

$$\xi = -f(x_0)/f'(x_0) \tag{56}$$

or namely at $x_1 = x_0 + \xi$:

$$x_1 = x_0 - f(x_0)/f'(x_0) \tag{57}$$

Since (55) is only approximate, x_1 is also only approximate, but it is a better (usually much better) approximation than x_0. The process can be repeated

$$x_0 \mapsto x_1 \mapsto x_2 \mapsto \dots \quad (58)$$

which converges rapidly to the correct solution. **Figure 1** illustrates the geometric interpretation of each step of the map (58).

5.2 Some applications

Example 3
Solve the equation $\sin x = x/2$ for x in radians. Put

$$f(x) = \sin x - x/2 \quad (59)$$

A rough sketch and the following tabulation gives an initial guess: $x_0 = 1.9$ is probably a good guess to start with.

x	$\sin x$	$x/2$	f
$\pi/2 = 1.57$	1.00	0.78	0.22
2	0.91	1.0	-0.09

Table 1. Tabulating the function $f(x)$

Iteration converges rapidly, as shown in the table below.

x	f
1.9	-3.7×10^{-3}
1.895506	-9.6×10^{-6}
1.895394	-6.5×10^{-11}

Table 2. Newton iteration

Problem 16
The intensity I per unit wavelength interval of black body radiation at temperature T is given by

$$I = I_0 \frac{x^5}{e^x - 1} \quad (60)$$

where I_0 is a constant and x is the dimensionless variable

$$x = \frac{hc}{\lambda k_B T} \quad (61)$$

where h is Planck's constant, c is the velocity of light, λ is the wavelength, T is the temperature and k_B is Boltzmann's constant.[3] To find the wavelength where the intensity is maximum, we seek the maximum of

$$g(x) = \frac{x^5}{e^x - 1} \quad (62)$$

So we set

$$0 = g'(x) = \frac{5x^4}{e^x - 1} - \frac{x^5}{(e^x - 1)^2} \cdot e^x \quad (63)$$

This is the same as seeking the root of

$$f(x) = x - 5 + 5e^{-x} \quad (64)$$

An initial guess is $x \approx 5$ (since for x this large, the exponential term is quite small).
(a) Use Newton's method to find the value of x to 3 significant figures.
(b) If cosmic microwave background has a temperature of 2.75 K, find the peak wavelength in mm.
§

Appendix

A Convergence

Here we give a simple criterion for convergence of a power series.

Ratio test
From Problem 1, we see that the geometric series $\sum_n r^n$ converges if $0 \le r < 1$. Thus any series $\sum_n T_n$ (which does not even have to be a power series) will also converge if its terms decrease faster than the geometric series, i.e., if there is some r such that $0 \le r < 1$ and some N, such that for any $n > N$,

$$\left| \frac{T_{n+1}}{T_n} \right| \le r \quad (65)$$

for all $n > N$. We can exclude the first N terms because any finite number of terms do not affect convergence.

Exponential function
Applied to the series for the exponential function

[3] See the module on *Dimensional analysis: Part II* where this function is discussed and the peak identified in a less efficient way.

for a given x, the ratio is

$$\frac{|x|}{n+1} \tag{66}$$

If we choose N to be some integer larger than $|x|$, then the ratio test is satisfied. Thus the series for the exponential function converges for any value of the argument.

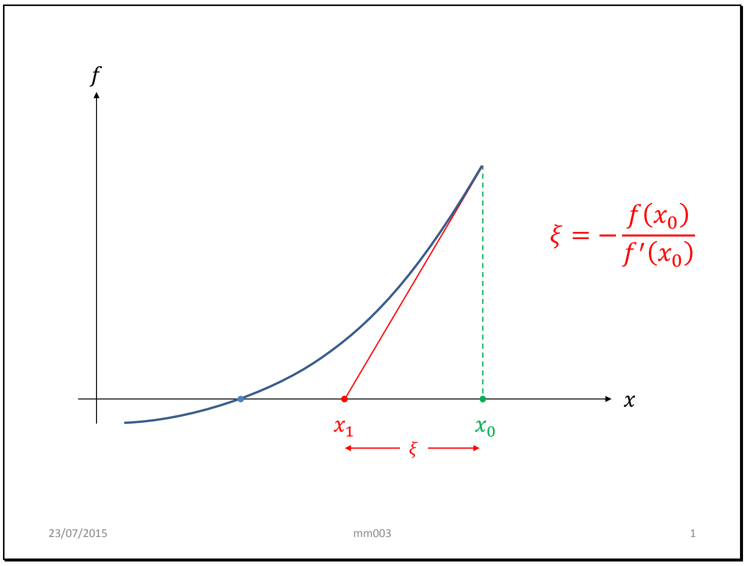

Figure 1: Newton's method