S.Q. 24.

Aim: To calculate the laplacian in plane polar coordinate

When the potential is circular symmetric, you are suggested to expressed the Hamiltonian in plane polar coordinates. $(\hat{H}(x, y, \frac{\partial}{\partial x}, \frac{\partial}{\partial y})$

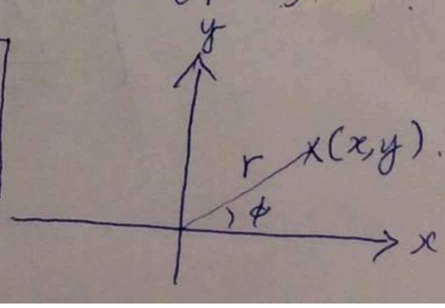
$\rightarrow \hat{H}(r, \phi, \frac{\partial}{\partial r}, \frac{\partial}{\partial \phi})$. It's because it's usually way more easier to do the problem in this way.

In this question, we want to express $\vec{\nabla}^2_{2D} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, relating to kinetic energy, in terms of $r, \phi, \frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \phi}$.

Proof: $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

Step ①: Express x, y in terms of r, ϕ.

$$x = r\cos\phi \quad — \boxed{1}$$
$$y = r\sin\phi \quad — \boxed{2}$$



Step ②: Express r, ϕ in terms of x, y

$$r = \sqrt{x^2+y^2} \quad - \boxed{3}$$

$$\tan\phi = \frac{y}{x} \quad - \boxed{4}$$

Step ③ Find the chain rules for $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$.

If the wavefunction is expressed in plane polar coordinate : $\varphi(r(x,y), \phi(x,y))$.

Then,
$$\begin{cases} \dfrac{\partial\varphi}{\partial x} = \dfrac{\partial\varphi}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial\varphi}{\partial\phi}\dfrac{\partial\phi}{\partial x} & (*) \\ \dfrac{\partial\varphi}{\partial y} = \dfrac{\partial\varphi}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial\varphi}{\partial\phi}\dfrac{\partial\phi}{\partial y} & (\star) \end{cases}$$

Using $\boxed{3}$:
$$\begin{cases} \boxed{\dfrac{\partial r}{\partial x} = \dfrac{2x}{2\sqrt{x^2+y^2}} = \dfrac{x}{r} = \cos\phi} \\ \boxed{\dfrac{\partial r}{\partial y} = \dfrac{2y}{2\sqrt{x^2+y^2}} = \dfrac{y}{r} = \sin\phi} \end{cases}$$

Using $\boxed{4}$:
$$\begin{cases} \dfrac{\partial}{\partial x}\tan\phi = \dfrac{\partial}{\partial x}\left(\dfrac{y}{x}\right) \\ \dfrac{\partial\phi}{\partial x}\sec^2\phi = -\dfrac{y}{x^2} = -\dfrac{r\sin\phi}{r^2\cos\phi} \\ \boxed{\dfrac{\partial\phi}{\partial x} = -\dfrac{1}{r}\sin\phi} \\ \dfrac{\partial}{\partial}\tan\phi = \dfrac{\partial}{\partial y}\left(\dfrac{y}{x}\right) \\ \dfrac{\partial\phi}{\partial y}\sec^2\phi = \dfrac{1}{x} = \dfrac{1}{r\cos\phi} \\ \boxed{\dfrac{\partial\phi}{\partial y} = \dfrac{1}{r}\cos\phi} \end{cases}$$

Therefore, the chain rules become. (P.3)

Using (*) and (☆)

$$\left(\frac{\partial}{\partial x}\right)\varphi = \left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}\right)\varphi.$$

$$= \left(\cos\phi \frac{\partial}{\partial r} - \frac{1}{r}\sin\phi \frac{\partial}{\partial \phi}\right)\varphi$$

$$\left(\frac{\partial}{\partial y}\right)\varphi = \left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}\right)\varphi$$

$$= \left(\sin\phi \frac{\partial}{\partial r} + \frac{1}{r}\cos\phi \frac{\partial}{\partial \phi}\right)\varphi.$$

Step 14, you can calculate $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial y^2}$.

$$\frac{\partial^2}{\partial x^2}\varphi = \left(\cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial}{\partial \phi}\right)\left(\cos\phi\frac{\partial \varphi}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial \varphi}{\partial \phi}\right)$$

$$= \cos\phi\frac{\partial}{\partial r}\left(\cos\phi\frac{\partial \varphi}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial \varphi}{\partial \phi}\right)$$
$$- \frac{1}{r}\sin\phi\frac{\partial}{\partial \phi}\left(\cos\phi\frac{\partial \varphi}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial \varphi}{\partial \phi}\right).$$

$$= \cos\phi\left(\cos\phi\frac{\partial^2\varphi}{\partial r^2} + \sin\phi\frac{1}{r^2}\frac{\partial\varphi}{\partial\phi} - \sin\phi\frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\phi}\right)$$
$$- \frac{1}{r}\sin\phi\left(-\sin\phi\frac{\partial\varphi}{\partial r} + \cos\phi\frac{\partial^2\varphi}{\partial r\partial\phi} - \cos\phi\frac{1}{r}\frac{\partial\varphi}{\partial\phi} - \sin\phi\frac{1}{r}\frac{\partial^2\varphi}{\partial\phi^2}\right).$$

$$= \cos^2\phi\frac{\partial^2\varphi}{\partial r^2} + \sin^2\phi\frac{1}{r}\frac{\partial\varphi}{\partial r} - 2\sin\phi\cos\phi\frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\phi} + 2\sin\phi\cos\phi\frac{1}{r^2}\frac{\partial\varphi}{\partial\phi}$$
$$+ \sin^2\phi\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\phi^2}$$

$$\left(\frac{\partial^2}{\partial y^2}\right)\varphi = \left(\sin\phi\frac{\partial}{\partial r} + \cos\phi\frac{1}{r}\frac{\partial}{\partial\phi}\right)\left(\sin\phi\frac{\partial}{\partial r} + \cos\phi\frac{1}{r}\frac{\partial}{\partial\phi}\right)$$

$$= \sin\phi\frac{\partial}{\partial r}\left(\sin\phi\frac{\partial\varphi}{\partial r} + \cos\phi\frac{1}{r}\frac{\partial\varphi}{\partial\phi}\right)$$
$$+ \frac{1}{r}\cos\phi\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial\varphi}{\partial r} + \cos\phi\frac{1}{r}\frac{\partial\varphi}{\partial\phi}\right)$$

$$= \sin\phi\left(\sin\phi\frac{\partial^2\varphi}{\partial r^2} - \cos\phi\frac{1}{r^2}\frac{\partial\varphi}{\partial\phi} + \cos\phi\frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\phi}\right)$$
$$+ \cos\phi\frac{1}{r}\left(\cos\phi\frac{\partial\varphi}{\partial r} + \sin\phi\frac{\partial^2\varphi}{\partial r\partial\phi} - \sin\phi\frac{1}{r}\frac{\partial\varphi}{\partial\phi} + \cos\phi\frac{1}{r}\frac{\partial^2\varphi}{\partial\phi^2}\right)$$

$$= \sin^2\phi\frac{\partial^2\varphi}{\partial r^2} + \cos^2\phi\frac{1}{r}\frac{\partial\varphi}{\partial r} + 2\sin\phi\cos\phi\frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\phi} - 2\sin\phi\cos\phi\frac{1}{r^2}\frac{\partial\varphi}{\partial\phi}$$
$$+ \cos^2\phi\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\phi^2}.$$

Therefore,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\varphi$$

$$=\left(\frac{\partial^2}{\partial x^2}\right)\varphi+\left(\frac{\partial^2}{\partial y^2}\right)\varphi$$

$$=\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\phi^2}$$

$$=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right)\varphi$$

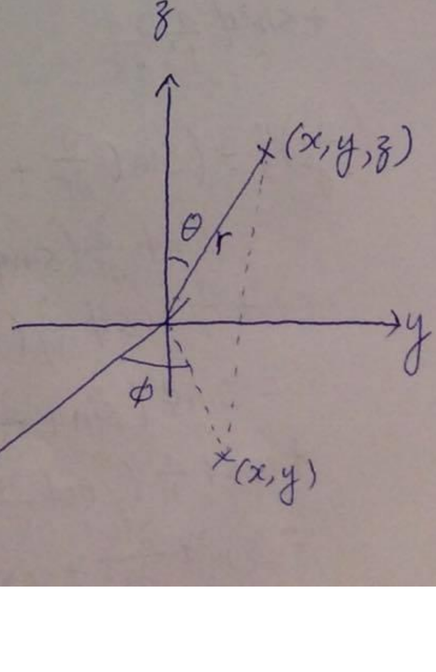
S.Q. 25

> Aim: To demonstrate that a state of definite L_z has uncertain $\hat{L}_y$ and $\langle\hat{L}_y\rangle=0$

(a) We want to express $\hat{L}_y$ in spherical coordinates, starting with $\left(\hat{L}_y=\hat{z}\hat{p}_x-\hat{x}\hat{p}_z=\frac{\hbar}{i}\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)\right)$

Step ① Express x, y, z in terms of r, θ, ϕ.

$$x=r\cos\theta\cos\phi \quad — \boxed{1}$$
$$y=r\sin\theta\sin\phi \quad — \boxed{2}$$
$$z=r\cos\theta \quad — \boxed{3}$$



Step ② Express r, θ, ϕ in terms of x, y, z

$$r = \sqrt{x^2+y^2+z^2} \quad - \boxed{4}$$

$$\cos\theta = \frac{z}{r} \quad - \boxed{5}$$

$$\tan\phi = \frac{y}{x}. \quad - \boxed{6}$$

Step ③ Find the chain rules for $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial z}$.

If the wavefunction is expressed in polar coordinates: $\varphi(r(x,y,z), \theta(x,y,z), \phi(x,y,z))$

Then,
$$\begin{cases} \frac{\partial \varphi}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial \varphi}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}. \\ \frac{\partial \varphi}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial \varphi}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial \phi}. \end{cases}$$

Using $\boxed{4}$:
$$\begin{cases} \boxed{\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r} = \sin\theta\cos\phi} \\ \boxed{\frac{\partial r}{\partial z} = \frac{2z}{2\sqrt{x^2+y^2+z^2}} = \frac{z}{r} = \cos\theta} \end{cases}$$

Using $\boxed{5}$:
$$\begin{cases} \frac{\partial}{\partial x}\cos\theta = \frac{\partial}{\partial x}\left(\frac{z}{r}\right) \\ (-\sin\theta)\frac{\partial \theta}{\partial x} = z\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x} \\ (-\sin\theta)\frac{\partial \theta}{\partial x} = -\frac{zx}{r^3} \\ (-\sin\theta)\frac{\partial \theta}{\partial x} = -\frac{(\cos\theta)(\sin\theta\cos\phi)}{r} \\ \boxed{\frac{\partial \theta}{\partial x} = \frac{\cos\theta\cos\phi}{r}} \\ \frac{\partial}{\partial z}\cos\theta = \frac{\partial}{\partial z}\left(\frac{z}{r}\right) \\ (-\sin\theta)\frac{\partial \theta}{\partial z} = \frac{1}{r} - \frac{z}{r^2}\frac{\partial r}{\partial z} \end{cases}$$

$$(-\sin\theta)\left(\frac{\partial\theta}{\partial z}\right) = \frac{1}{r} - \frac{z^2}{r^3}$$
$$= \frac{1}{r} - \frac{\cos^2\theta}{r}$$
$$= \frac{\sin^2\theta}{r}$$
$$\boxed{\frac{\partial\theta}{\partial z} = -\frac{\sin\theta}{r}}$$

Using $\boxed{6}$:

$$\frac{\partial}{\partial x}\tan\phi = \frac{\partial}{\partial x}\left(\frac{y}{x}\right).$$
$$\frac{\partial\phi}{\partial x}\sec^2\phi = -\frac{y}{x^2}$$
$$= -\frac{\sin\phi}{r\sin\theta\cos^2\phi}$$
$$\boxed{\frac{\partial\phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta}}$$
$$\frac{\partial}{\partial z}\tan\phi = \frac{\partial}{\partial z}\left(\frac{y}{x}\right)$$
$$\frac{\partial\phi}{\partial z}\sec^2\phi = 0$$
$$\boxed{\frac{\partial\phi}{\partial z} = 0}$$

$$\therefore \quad \frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial\phi}$$
$$= \sin\theta\cos\phi\frac{\partial}{\partial r} + \cos\theta\cos\phi\frac{1}{r}\frac{\partial}{\partial\theta} - \sin\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$
$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial z}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial\phi}$$
$$= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}.$$

$$z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} = (r\cos\theta)\frac{\partial}{\partial x} - (r\sin\theta\cos\phi)\frac{\partial}{\partial z}.$$
$$= \left(r\sin\theta\cos\theta\cos\phi\frac{\partial}{\partial r} + \cos^2\theta\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$
$$- \left(r\sin\theta\cos\theta\cos\phi\frac{\partial}{\partial r} - \sin^2\theta\cos\phi\frac{\partial}{\partial\theta}\right)$$
$$= \cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}.$$

$\therefore \hat{L}_y = \frac{\hbar}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$

$= \frac{\hbar}{i}\left(\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)$

(b). In this question, we want to calculate $\hat{L}_y Y_{11}(\theta,\phi)$.

$$Y_{11}(\theta,\phi) = -\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{i\phi}$$

$$\hat{L}_y Y_{11}(\theta,\phi) = -\left(\frac{3}{8\pi}\right)^{1/2}\left(\frac{\hbar}{i}\right)\left(\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)(\sin\theta e^{i\phi})$$

$$= -\left(\frac{3}{8\pi}\right)^{1/2}\left(\frac{\hbar}{i}\right)\left(\cos\phi\cos\theta e^{i\phi} - \sin\phi\cot\theta\sin\theta(i)e^{i\phi}\right)$$

$$= -\left(\frac{3}{8\pi}\right)^{1/2}\left(\frac{\hbar}{i}\right)\left(\cos\phi\cos\theta e^{i\phi} - i\sin\phi\cos\theta e^{i\phi}\right)$$

$(e^{i\phi} = \cos\phi + i\sin\phi)$

$$= -\left(\frac{3}{8\pi}\right)^{1/2}\left(\frac{\hbar}{i}\right)\left(\cos^2\phi\cos\theta + i\cos\phi\sin\phi\cos\theta - i\sin\phi\cos\theta\cos\phi + \sin^2\phi\cos\theta\right)$$

$$= -\left(\frac{3}{8\pi}\right)^{1/2}\left(\frac{\hbar}{i}\right)\cos\theta$$

$$= -\frac{1}{\sqrt{2}}\left(\frac{\hbar}{i}\right)\left[\left(\frac{3}{4\pi}\right)^{1/2}\cos\theta\right]$$

$$= -\frac{1}{\sqrt{2}}\left(\frac{\hbar}{i}\right)Y_{10}(\theta,\phi).$$

Therefore $Y_{11}(\theta,\phi)$ is not an eigenfunction of $\hat{L}_y$.

(c). In this question, we want to demonstrate that a state of definite L_z ($Y_{11}(\theta,\phi)$) has ① uncertain $\hat{L}_y$ and ② $\langle\hat{L}_y\rangle = 0$.

① Uncertain $\hat{L}_y$.

Since $[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x \neq 0$

$\Rightarrow$ A state of definte should have uncertain L_y.

② $\langle L_y \rangle = 0$.

$$\int Y_{11}(\theta,\phi)\, \hat{L}_y\, Y_{11}(\theta,\phi)\, d\Omega$$

$$= -\frac{1}{\sqrt{2}}\left(\frac{\hbar}{i}\right) \int Y_{11}(\theta,\phi)\, Y_{10}(\theta,\phi)\, d\Omega$$ (using the result in (b))

$$= 0.$$

Therefore, $Y_{11}(\theta,\phi)$ has uncertain $\hat{L}_y$ and $\langle \hat{L}_y \rangle = 0$.

SQ26)

H.O. potential in cartesian coordinate:

$U(x,y) = \frac{1}{2} m\omega^2 (x^2+y^2)$

In plane polar coordinates:

$U(r,\phi) = U(r) = \frac{1}{2} m\omega^2 r^2$, here we only consider $U(r)$

For $E = 2\hbar\omega$, we know that in cartesian coordinate the possible eigenstates are $(n_x, n_y) = (1,0)$ or $(0,1)$ $\because E_{n_x,n_y} = (n_x+n_y+1)\hbar\omega$

For $(n_x, n_y) = (1,0)$,

$$\psi_{10}(x,y) = \psi_1(x)\psi_0(y) = \left[\left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} \sqrt{2a}\, x\, e^{-ax^2/2}\right]\left[\left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-ay^2/2}\right]$$

$$= \alpha \sqrt{\frac{2}{\pi}}\, x\, e^{-\frac{a}{2}(x^2+y^2)}$$

For $(n_y, n_x) = (0,1)$, similarly we get

$$\psi_{01}(x,y) = \psi_0(x)\psi_1(y) = \alpha\sqrt{\frac{2}{\pi}}\, y\, e^{-\frac{a}{2}(x^2+y^2)}$$

Form the new eigenstate as $\psi_\pm(x,y) = \frac{1}{\sqrt{2}}[\psi_1(x)\psi_0(y) \pm i\psi_0(x)\psi_1(y)]$!

$$\psi_\pm(x,y) = \frac{1}{\sqrt{2}}\left[\alpha\sqrt{\frac{2}{\pi}}\, x\, e^{-\frac{a}{2}(x^2+y^2)} \pm i\,\alpha\sqrt{\frac{2}{\pi}}\, y\, e^{-\frac{a}{2}(x^2+y^2)}\right]$$

$$= \frac{\alpha}{\sqrt{\pi}} e^{-\frac{a}{2}(x^2+y^2)} (x \pm iy)$$

$\because x = r\cos\phi$, $y = r\sin\phi$, $x^2+y^2 = r^2$

$$\therefore \psi_\pm(r,\phi) = \frac{\alpha}{\sqrt{\pi}} e^{-\frac{a}{2}r^2} (r\cos\phi \pm ir\sin\phi)$$

$$= \frac{\alpha}{\sqrt{\pi}}\, r e^{-\frac{a}{2}r^2} (\cos\phi \pm i\sin\phi)$$

* Euler formula: $e^{\pm i\phi} = \cos\phi \pm i\sin\phi$!

$$\therefore \psi_+(r,\phi) = \frac{\alpha}{\sqrt{\pi}}\, r e^{-\frac{a}{2}r^2} e^{i\phi}$$

$$\psi_-(r,\phi) = \frac{\alpha}{\sqrt{\pi}}\, r e^{-\frac{a}{2}r^2} e^{-i\phi}$$

They are in the form $\psi(r,\phi) = R(r)\Phi(\phi)$!