PHYS4031 STATISTICAL MECHANICS

SAMPLE QUESTION FOR DISCUSSION IN WEEK 8 (26 October 2016) and Week 9 (31 October 2016) EXERCISE CLASSES

You may want to think about them before attending exercise class.
SQ19: The density of states (Ch.VIII) for arbitrary spatial dimensions and dispersion relations.
SQ20: Density of normal modes $D(\omega)$ of Debye model in 3D/2D and for light (black-body radiation formula).
SQ21: Carbon monoxide - real gas, ideal gas, van der Waals gas
SQ22: (Not for discussion in Exercise Classes) Extra SQ on Two-particle wavefunctions for two non-interacting fermions/bosons. It was a homework in PHYS3022 (Applied Quantum Mechanics).

MID-TERM EXAMINATION ANNOUNCEMENT - NOTE NEW VENUE!

Date: 5 November 2016 (Saturday)
Time: 10:00AM (2-hour exam)
Venue: LSK 304 (Lee Shau Kee Building)
Coverage: Everything from Ch.1 to Ch.9 Part 1 (inclusive), including materials discussed in class notes, lectures, sample questions, and problem sets.

SQ19 **Density of single-particle states (Ch.VIII) - "One problem fits all"**

This is **one problem** on the topic of density of single-particle states $g(\epsilon)$ that **covers many problems**. The answer can also be applied to similar problems in counting the number of normal mode (angular) frequencies $D(\omega)$ where $D(\omega)d\omega$ is the number of normal modes with angular frequencies in the interval ω to $\omega + d\omega$.

Consider particles of spin s and thus a spin-degeneracy of $G_s = 2s + 1$. Work out the density of states, i.e., density of single-particle states, in d-dimension, given that the energy dispersion relation is $\epsilon(\mathbf{k}) = \epsilon(k) = Ak^{\alpha}$, where $k = |\mathbf{k}|$ the magnitude of the wavevector. The pre-factor A, of course, will be different in different situations so as to make the units right.

[Students: In physics, we don't want to spend time on working out specific cases. Instead, we like "generality" and "universal approaches". Here is a problem that covers many cases. Step 1, identify the dimensionality of the problem and work in k-space. Then in your mind imagine two cuts at ϵ and $\epsilon + d\epsilon$. The states in the interval must fall onto a region in k-space. Step 2, changing from the variable k to the variable ϵ by feeding in the information of the energy dispersion relation $\epsilon(\mathbf{k})$. In particular, this SQ covers non-relativistic particles, free electrons in metals, phonons (lattice vibrations in solids, thus heat capacity) in long-wavelength limit, photons (black-body formula, see SQ20), surface waves, sound waves in liquid helium, and ultra-relativistic particles in any spatial dimensions. Cool!]

TA: Try the following **two methods**.

(a) Do it in the standard way of "two cuts" between ϵ to $\epsilon + d\epsilon$ and relate the number of states to k to $k + dk$, as described above.

(b) Also try to get the answer by first considering the number of states with energy less than ϵ (thus the states within a certain part of a "sphere" in k-space) and then take a derivative. Of course, the answer should come out to be the same. We used the same argument in first getting $W^{<}(E, V, N)$ and then $\mathcal{W}(E, V, N) = \partial W^{<}/\partial E$ in the microcanonical ensemble classical ideal gas example in Week 2.

SQ20 **Density of Normal Modes of Debye model in 3D and 2D solids and Black-body Radiation Formula**

This SQ fills in two pieces of information that we used previously. In Ch.VI, we discussed how Debye came in to correct Einstein's model and got the observed behavior of the heat capacity $C \sim T^3$ at low temperatures. There, we invoked that $D_{3D}(\omega) \sim \omega^2$ for the number of normal modes in a 3D solid for the low frequency normal modes. In SQ16, we invoked $D_{2D}(\omega) \sim \omega$ for a 2D solid and argued that fluctuations will destroy the ordered phase at any finite temperature in 2D.

(a) Within the spirit of the Debye model, we take the linear part at low frequencies and write it as $\omega(k) = v_s k$, where v_s is a speed related to the speed of sound in a solid. Apply the results in SQ19 to show that ω-dependence of $D_{3D}(\omega)$ and $D_{2D}(\omega)$ for a 3D solid and a 2D solid, respectively.

(b) **Thermal (Black-body) radiation formula.** Here is a twist. Let's consider light in a 3D box. Since light follows $\omega = ck$, it is of the "Debye" form. So $D_{3D}(\omega)$ follows. (Perhaps needs to count the polarizations.) Next, EM waves of a certain angular frequency refers to oscillating $\mathbf{E}$ and $\mathbf{B}$ fields. Thus it is not surprising to see that we can use oscillator's statistical physics to treat excitation of light (photons). At a temperature T, there are $1/(e^{\beta\hbar\omega} - 1)$ photons of energy $\hbar\omega$ in the box. Of course, there are many possible ω's.

TA: Hence, construct the photon energy at temperature T for angular frequencies between ω to $\omega + d\omega$. Hence, construct the black-body radiation formula (usually given as energy per area per second) which is the energy flux (thus needs a factor of c or more precisely $c/4$) per unit volume. You may first get the result in ω and then turn it to other forms (given in textbooks) in terms of frequency ν or wavelength λ.

SQ21 **CO: Real gas versus ideal gas and van der Waals gas**

Given that the experimental value of the molar volume v of CO (carbon monoxide) is 0.04009 L·mol^{-1} at a temperature 200 K and a pressure 1000 bar.

(a) Taking the gas as an ideal gas, calculate the molar volume at the given temperature and pressure.

(b) Taking the van der Waals gas law, i.e.,

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT \ , \tag{1}$$

calculate the molar volume at the given temperature and pressure. The parameters are $a = 1.4734$ dm^6 ·bar·mol^{-2} and $b = 0.039523$ dm^3·mol^{-1} for CO.

Hence, compare the results in (a) and (b) with the experimental value. [Hint: Be careful of units. It may be useful to first express the gas constant R into the right units for the problem.]

SQ22 was a homework problem in Applied Quantum Mechanics (PHYS3022). We will discuss ideal Fermi/Bose gas soon. This SQ serves to illustrate that even at the level of considering two non-interacting particles, fermions and bosons give very different results. The difference illustrates that the quantum nature of the particles really matters! As some students did not go through PHYS3022, it is put here for completeness. For those who took PHYS3022 but forgot what we did, this serves as a reminder.

SQ22 **Forming Two-particle states from single-particle states.**

This is an important problem that every (CUHK) physics student must do. The results will also be used in Statistical Mechanics. It is related to the simplest QM problem – 1D particle-in-a-box.

(a) Consider a particle of mass m confined in a 1D infinite well of size L. [No derivation needed.] Write down the normalized energy eigenstates $\psi_n(x)$ and the corresponding energy eigenvalues E_n. These are the single-particle states in this problem.

(b) **Forming Two-particle states.** Consider **two distinguishable particles** of the same mass in a 1D well. The particles are non-interacting (e.g., they are chargeless). Given that particle 1 is in ψ_m and particle 2 is in ψ_n, write down a two-particle wavefunction $\psi_{dist}(x_1, x_2)$. What is the corresponding energy eigenvalue? Find the probability that both particles are in the left hand side of the well, i.e., the probability that a measurement shows $0 < x_1 < L/2$ *and* $0 < x_2 < L/2$?

(c) Similar to (b), but now the **two particles are indistinguishable (identical)**. **Don't worry about the spin part of the total wavefunction for the moment and concentrate on the spatial part.** Write down a two-particle wavefunction $\psi_{sym}(x_1, x_2)$ that is **symmetric** with respect to interchanging the coordinates x_1 and x_2 of the two particles. Hence, find the probability that both particles are in the left hand side of the well, i.e., the probability that a measurement shows $0 < x_1 < L/2$ *and* $0 < x_2 < L/2$? How is your result compared with that in (b)? This is the bosonic case.

(d) Similar to (b), but now the two particles are indistinguishable (identical). Again, don't worry about the spin part of the total wavefunction for the moment and concentrate on the spatial part. Write down a two-particle wavefunction $\psi_{anti}(x_1, x_2)$ that is **anti-symmetric** with respect to interchanging the coordinates x_1 and x_2 of the two particles. Hence, find the probability that both particles are in the left hand side of the well, i.e., the probability that a measurement shows $0 < x_1 < L/2$ *and* $0 < x_2 < L/2$? How is your result compared with that in (b) and (c)? This is the fermionic case.

(e) Discussion – Which symmetry has an enhanced probability (relative to the case of distinguishable particles) of finding the two particles on the same side? Which symmetry has an reduced probability of finding the two particles on the same side? [**Read me:** Note that the particles are **non-interacting**, but still the symmetry requirement has an effect that seems to push the particles closer OR to pull the particles farther apart. It looks as if the "exchange symmetry requirement" leads to a "force" between the particles (either attraction or repulsion). This is called the exchange force and it plays a fundamental role in many phenomena, e.g., the origin of magnetism, when we include spin wavefunction into consideration.]

(f) A picture is worth a thousand words. Let's say one particle is in the single-particle ground state $m = 1$ (thus ψ_1) and another in the 1st excited state $n = 2$ (thus ψ_2). Now use your favorite graphic software and plot $\psi_{dist}(x_1, x_2)$, $\psi_{sym}(x_1, x_2)$, and $\psi_{anti}(x_1, x_2)$ as a function of x_1 and x_2 for the range $0 < x_1 < L$ and $0 < x_2 < L$. Plot also $|\psi_{dist}(x_1, x_2)|^2$, $|\psi_{sym}(x_1, x_2)|^2$, and $|\psi_{anti}(x_1, x_2)|^2$ as a function of x_1 and x_2 for the range $0 < x_1 < L$ and $0 < x_2 < L$. Note that these are 3D plots, as one axis is x_1, another is x_2 and the third axis is the quantity you want to plot. At this point (see part (g)), look at the plots carefully and comment on the difference between $|\psi_{sym}(x_1, x_2)|^2$ and $|\psi_{anti}(x_1, x_2)|^2$.

(g) (Important physics! Read me.)

Now, comparing the plots in (f) for two fermions and two bosons, we see the two identical non-interacting bosons tend to come close to each other and two identical non-interacting fermions tend to avoid each other. Note that the two particles are **non-interacting**, i.e., there are no attractive or repulsive forces (Coulombic or gravitation, etc.) between them. Yet, when we account for **the symmetric requirement due to the quantum nature of the two particles properly**, there seems to be some kind of **an effective interaction** between them! It is exactly why even non-interacting Fermi and Bose gases behave very differently from classical gas, when T is low and/or concentration is high. For the same reason, we do NOT expect to see $pV = NkT$ any more for ideal Fermi gas and ideal Bose gas, even when the temperature is not too low. Instead, we expect to see some correction terms such as $pV/NkT = 1 +$ *(something)* at "high, but not-too-high" temperatures for Fermi and Bose gases, with (something) bring the correction to ideal gas behavior due to the fermionic/bosonic nature of the particles. This is indeed the case, and we will work that (something) later.