PHYS4031 STATISTICAL MECHANICS

SAMPLE QUESTION FOR DISCUSSION IN WEEK 13 EXERCISE CLASSES (28 Nov and 30 Nov 2016)
You may want to think about them before attending exercise class.

PHYS4031 ANNOUNCEMENT ON FINAL EXAM

Coverage: Chapter I to Chapter XIII (end of Ideal Fermi Gas chapter), including all materials discussed in class notes, lectures, sample questions in exercise classes, and problem sets. Sections and appendices in class notes marked "Optional" are excluded.
Time/Venue: Arranged centrally by University. Check time/venue yourself.

SQ30 - $T = 0$ physics of 3D ultra-relativistic fermions
SQ31 - A summary of the course

SQ30 $T = 0$ **physics of 3D ultra-relativistic fermions.**

For non-interacting fermions, the general equations for N and E are

$$N = \sum_{s.p.\ states\ i} \frac{1}{e^{(\epsilon_i-\mu)/kT}+1} = \int g(\epsilon)\frac{1}{e^{(\epsilon-\mu)/kT}+1}\, d\epsilon \tag{1}$$

$$E = \sum_{s.p.\ states\ i} \frac{\epsilon_i}{e^{(\epsilon_i-\mu)/kT}+1} = \int g(\epsilon)\frac{\epsilon}{e^{(\epsilon-\mu)/kT}+1}\, d\epsilon \tag{2}$$

where the summations are over single-particle states and $g(\epsilon)$ is the density of states. We also have the additional equation of $pV = -\Omega = kT \ln Q$ that helps us jump into an equation of state. **These equations are general. They are good for non-interacting fermions in any spatial dimension and any energy dispersion relation.** Details of the system are hidden in the density of states $g(\epsilon)$.

We worked out the physics of a 3D *non-relativistic* Fermi gas in Chapter XIII. The results are applicable to metals and thus they form a part of solid state physics. In class, we mentioned that the degenerate pressure is responsible for opposing the gravitational pull (collapsing a star). However, the speeds of the fermions (electrons) in astrophysical objects often require us to consider relativistic effects. The full relativistic $\epsilon(p) = \sqrt{c^2p^2 + m^2c^4}$ is not easy to deal with. An easier situation is that of a 3D gas of ultra-relativistic fermions.

$T = 0$ **physics of a gas of 3D ultra-relativistic fermions** – Now, consider a gas of N ultra-relativistic and non-interacting spin-half fermions in a volume V. The energy dispersion relation is then modelled by $\epsilon(k) = c\hbar k \sim k$, where $k = |\mathbf{k}|$ is the magnitude of the wavevector.

(a) Find the density of states $g(\epsilon)$ and contrast it with that of 3D non-relativistic particles.

(b) Consider $T = 0$ and work out the $T = 0$ physics, including how E_F depends on the fermion number density N/V and how the $T = 0$ **pressure** depends on N/V.

[Remarks: The pressure $p \sim (N/V)^{5/3}$ for non-relativistic particles and $p \sim (N/V)^{4/3}$ for ultra-relativistic particles. The difference turns out to be important in astrophysical contexts, as the degenerate pressure ($T = 0$ pressure) needs to compete with the pressure due to gravitation.]

SQ31 **Looking at the course as a whole - A review on the Course.**

TA will give the key points that thread the different topics in the course as a quick review and collect what we have done as applications in class notes, sample questions and problem sets.

0. What is equilibrium statistical mechanics? Microscopic theory of thermodynamics. As such, it is a subject that handles a huge number of entities in a system.
1. "All accessible microstates are equally probable" and $S(E, V, N) = k \ln W(E, V, N)$
   What are the conditions under which the statement and equation hold? List the examples/applications based on this approach that we did.

2. Systems in equilibrium at a temperature T, with fixed T, V, N.

$$\begin{aligned} Z(T,V,N) &= \sum_{N\text{-particle states } i} e^{-\beta E_i} \\ &= \sum_{N\text{-particle energy level } i} W(E_i,V,N)\, e^{-\beta E_i} \\ &= \int \mathcal{W}(E,V,N)\, e^{-\beta E}\, dE \end{aligned} \tag{3}$$

What are being summed/integrated over and why are they equivalent?

$$F(T,V,N) = -kT \ln Z(T,V,N) \tag{4}$$

What is dF and how to get entropy, pressure, and chemical potential?

List the examples/applications that we did based on the partition function Z.

For N independent, distinguishable and identical 2-level systems, write Z.

For N independent, distinguishable and identical quantum oscillators, write Z.

3. Write $Z(T,V,N)$ for general classical statistical mechanical calculations. List problems/applications did in classical statistical mechanics.

   Write Z for classical ideal gas.

4. Interacting systems, phase transitions and critical phenomena.

   List examples of interacting systems did in the course.

   Phase diagram of a pure substance.

   Van der Waals equation of states describes the vapor-liquid transition. Role of interaction. The second virial coefficient and the physics behind the sign of $B_2(T)$.

   Critical point and behavior near the critical point. Phase diagram of Ising model and its similarity to the vapor-liquid diagram. Mean field theory and its results. Critical exponents.

5. Open systems in equilibrium at a temperature T and a chemical potential μ, thus systems with given T, V, μ.

$$Q(T,V,\mu) = \sum_{N=0}^{\infty} \sum_{N\text{-particle states } i} e^{-\beta E_i(N)+\beta\mu N} \tag{5}$$

$$\Omega(T,V,\mu) = -kT \ln Q(T,V,\mu) \tag{6}$$

What is $d\Omega$ and how to get entropy, pressure, and number of particles? Eq. (6) gives a short cut to the equation of state, what is it?

List problems did using the grand partition function.

6. Point out the physical meaning of

$$f_{FD}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}+1} \tag{7}$$

and

$$f_{BE}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)}-1} \tag{8}$$

Point out that these distributions can be obtained either as the most probable distributions (Lagrange multipliers method) or through the grand partition function $Q(T,V,\mu)$ for non-interacting fermions and bosons.

7. Point out the pieces of information (a) dimension of system, (b) dispersion relation, and (c) possible spin degeneracy, are embedded in the density of (single-particle) states $g(\epsilon)$. The density of states $g(\epsilon)$ has nothing to do with temperature. An example is $g(\epsilon)$ for 3D non-relativistic particles in volume V:

$$g(\epsilon) = G_s \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} \epsilon^{1/2} \tag{9}$$

List examples did on $g(\epsilon)$.

8. Taking 3D Ideal Fermi Gas as an example, use item 6 and item 7 to write down the equations for N and E. Point out that the equation of N serves to determine $\mu(T)$ for given N/V (number density of particles), and the equation of E gives $E(T)$ and thus $C_V(T)$.
9. Ideal Fermi Gas. Pauli Exclusion Principle dominates. Key features in $T = 0$ physics. Corrections of the order of $(kT/E_F)^2$ in low-temperature physics. Correction to classical gas behavior at high temperatures.
10. (For completeness: TA doesn't need to discuss this as it will be done in class in Week 13.) Ideal Bose Gas. Physics for the occurrence of Bose-Einstein condensation. T_c and macroscopic occupation of single-particle ground state. Experimental realization. Correction to classical gas behavior at high temperatures.
11. Math Skills: Gaussian Integrals, Gamma Function, Counting, Legendre Transforms, Stirling Formula, Infinite and finite sums, Taylor expansion, log and exponential, partial derivatives, set up self-consistent equation, Sommerfeld expansion, making approximations

This ends PHYS4031. If you follow the progress, your statistical mechanics knowledge allows you to start a postgraduate-level course in any good research university.

Further Reading. To consolidate your knowledge at about the **same level**, see

- F. Mandl, *Statistical Physics* 2nd ed. (John Wiley & Sons 1988) [traditional style]
- D. Yoshioka, *Statistical Physics: An Introduction* (Springer 2007) [concise and precise]
- C. Hermann, *Statistical Physics* (Springer 2005) [many applications to solid state problems]
- H. J. W. Muller-Kirsten, *Basics of Statistical Physics - A Bachelor Degree Introduction* (World Scientific 2010) [numerous worked-out examples]
- W. Greiner, L. Neise, and H. Stöcker, *Thermodynamics and Statistical Mechanics* (Springer 1995)
- S.J. Blundell and K.M. Blundell, *Concepts in Thermal Physics* (Oxford Univ. Press 2006) [comprehensive]

To move on to the **next level**, see

- D. Chandler, *Introduction to Modern Statistical Mechanics* (Oxford Univ. Press) [easier but brief]
- R.K. Pathria, *Statistical Mechanics* (2nd or later edition by Pathria and Beale) [we did first 8 chapters]
- M. Plischke and B. Bergersen, *Equilibrium Statistical Mechanics* (World Scientific 2006) [a solid treatment at graduate level]
- D.A. McQuarrie, *Statistical Mechanics* (University Science Books 2000) [we did 11/24 chapters, a book with topics on physical chemistry]