Educational Research Journal《教育研究學報》, Vol. 29, Nos. 1 & 2, 2014

Geometric Thinking Levels of Pre- and In-service Mathematics Teachers at Various Stages of Their Education

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This study investigated the geometric thinking levels (GTLs) of pre- and in-service mathematics teachers at various stages of their education in Israel. It focused on the first three GTLs according to van Hiele's theory. Furthermore, it examined whether there are differences in their mastery of GTLs in three main geometric topics.

The results indicate that the GTLs of pre-service teachers at their third and fourth years of education were similar to those in-service teachers studying for a master's degree in mathematics education. The GTLs of pre-service teachers in their first year and those of academics making a career change to mathematics teachers were the lowest. All the five participating groups demonstrated higher mastery of GTLs in triangles and quadrilaterals than in circles and three-dimensional geometric figures. In addition, regarding triangles and quadrilaterals, a large number of participants demonstrated that they had mastered the third level of geometric thinking. Fewer participants assimilated the third level in circles. All participants were not versed in the two higher levels regarding

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three-dimensional geometric figures. Most of them internalized only the first level or not even that level at all. The rest of them were diagnosed as "inconsistent" in mastering the GTLs.

Keywords: van-Hiele theory; geometric thinking levels; pre- and in-service mathematics teachers; academics making a career change to mathematics teachers

Theoretical Background

There are various theories dealing with the development of geometric thinking. One known theory was conceived by the Dutch wife and husband Dina van Hiele-Geldof and Pierre van Hiele (van Hiele, 1959, 1987). In 1959 the couple argued that there were five hierarchical levels. Due to doubts of mathematics educators, including van Hiele himself, as to the existence of the fifth level, it is customary today to relate only to four levels: recognition (naming) or visualization level, analysis or description level, ordering or informal deduction level, and rigor and deduction or formal deduction level (Burger & Shaughnessy, 1986; Gutiérrez, 1992; Patkin & Levenberg, 2010; van Hiele, 1987).

- *Level I: Recognition or visualization* — At this initial level, learners can identify geometric shapes and distinguish between them. Each of the concepts or the shapes is perceived as a whole, in the way it is seen. Learners are capable of distinguishing between similar shapes as well as naming them. At this level, learners are unable to specify the properties of those shapes.
- *Level II: Analysis or description* — At this level, learners can analyze properties of shapes but are unable to attribute properties of a particular item to the properties of the group to which it belongs.
- *Level III: Order or informal deduction* — At this level, learners identify a hierarchical order of connection between groups of different shapes according to their properties and definitions. However, they are incapable of proving claims related to the properties of the geometric shapes.

- *Level IV: Rigor and formal deduction* — At this level, learners understand the roles of basic concepts, axioms, definitions, theorems and proofs and their interrelations. They can use assumptions in order to prove theorems and understand the meaning of necessary and sufficient conditions. At this level, learners are able to provide reasons and arguments for the various levels of the proving process. Moreover, they comprehend the importance of discussing the proofs, the deduction from the particular to the general, and even the need for a proof of any kind.

Van Hiele presented five properties of the model: Sequential, advancement, intrinsic and extrinsic, linguistics, and mismatch (see Crowley, 1987).

- *Sequential* — Similarly to most models which engage in development, one should pass through the levels in a sequence. In order to succeed at a certain level, the strategies of previous levels should first be acquired.
- *Advancement* — Progress from one level to another depends more on contents and teaching methods than on age. The teaching methods should ascertain that learners do not skip or omit one level. Some of the teaching methods stimulate the progress and reinforce it whereas others delay or even prevent progress between two levels.
- *Intrinsic and extrinsic* — "The inherent objects at one level become the objects of study at the next level" (Crowley, 1987, p. 4); that is, a concept studied at a certain level "from above" and generally speaking becomes the topic of study at the next level. For example, at the first level of van Hiele's theory, the general matrix of the geometric shape is studied while at the second level, the shape is already defined according to its properties and components.
- *Linguistics* — Each level has its own linguistic symbols which characterize it. At the first level, the symbols are very simple and at higher levels, the symbols are more complicated. For example, a square is the simplest linguistic symbol assigned to a regular

quadrilateral. However, the name of a rectangle which is also a rhombus or of a rhombus which is also a rectangle matches the definition of a square. Only learners who are at the third level (order and informal deduction) can understand it while learners who are at lower levels encounter difficulties in understanding it.

- *Mismatch* — Learners who are at a certain level will find it difficult to understand contents and vocabulary typical of higher levels. Hence, it will be difficult for them to monitor the processes which transpire at the high level. In order to comprehend the contents and the processes of the higher level, one should first understand and master all the contents and all linguistic symbols typical of their level.

According to van Hiele's theory, partial mastery of a certain level is a prerequisite, though insufficient for mastering a higher level. People cannot be versed in a certain level before having mastered all its previous ones; otherwise, they are referred to as "inconsistent." Van Hiele's theory is related to plane geometry only. Some studies have recently applied the theory of plane geometry to other branches of mathematics, such as solid geometry (Patkin, 2010; Patkin & Sarfaty, 2012) and arithmetic (Crowley, 1987; Guberman, 2008). It is important to point out that most studies of difficulties of thinking levels focused generally on topics associated with plane geometry only, such as triangles and quadrilaterals (Halat & Şahin, 2008; Patkin, 1990; van Hiele, 1999) or engaged only in solid geometry (Gutiérrez, 1992; Patkin, 2010; Patkin & Sarfaty, 2012). There are no studies which relate to the mastery of thinking levels in different issues as a comprehensive picture and to comparison to mastery of thinking levels between the subjects.

Another characteristic of van Hiele's theory is that unlike other learning theories, particularly that of Piaget (1969/1975), the theory is grounded in the assumption that moving from one thinking level to another depends on teaching or learning experiences rather than on age or biological maturity (Geddes, Fuys, Lovett, & Tischler, 1982; van Hiele, 1999). Studies indicate that pupils encounter difficulties at every age (Koester, 2003) as do in-service and pre-service teachers (Halat & Şahin, 2008). A study of geometry and spatial thinking was conducted among kindergarten children and 6th graders who had to perform the same assignment. The findings showed that in spite of the age gap, there was only a minute difference in favor of the older children in performing the assignment (Clements & Battista, 1992). Another study explored "self-knowledge" of in-service elementary school mathematics teachers. It illustrated that while exposing their self-knowledge, in-service teachers manifested a lack of mastery and comprehension of solid geometry. However, after getting acquainted with van Hiele's theory, including experiencing, being in situations which encouraged reflection on reflection ("metacognition"), they progressed in their thinking levels, demonstrating openness and wish to learn, cope and improve (Patkin & Sarfaty, 2012). According to van Hiele's theory, based on the assumption that advancing from one thinking level to another is teaching-dependent, Crowley (1987) also argues that the activity type given to learners is meaningful. Her study investigating plane geometry thinking showed that the compliance between learners' level of comprehension and level of the assignments given to them is vital, if meaningful learning is to transpire.

The present study focused on the first three levels of van Hiele's theory. Assuming that progress in geometric thinking levels (according to van Hiele) is teaching-dependent, we deemed it appropriate to examine whether generally speaking there are differences in geometric thinking levels of elementary school mathematics teachers. That is, differences in the level of thinking between pre-service mathematics teachers in their first, third and fourth years of education, as compared to the level of thinking of in-service mathematics teachers studying toward a master's degree in mathematics education as well as academics making a career change to mathematics teachers.

Furthermore, we explored whether there are any differences in the participants' level of geometric thinking regarding three specific topics studied at elementary school (triangles and quadrilaterals, circles, and three-dimensional geometric figures).

Research Questions

1. Is there a difference in geometric thinking levels of pre-service and in-service mathematics teachers at different points during their education in geometry in general and in each of these three topics in particular: triangles and quadrilaterals, circles, and three-dimensional geometric figures?
2. At what thinking level do pre-service and in-service mathematics teachers at different points during their education master various topics of plane geometry (triangles and quadrilaterals, circles) and space geometry (three-dimensional geometric figures)?

Methodology

Research Population

The research population consisted of 142 in-service and pre-service mathematics teachers studying in an academic teacher education college. This population comprised different groups of participants: 46 pre-service teachers in their first year of education for becoming elementary school mathematics teachers; 30 pre-service teachers in their third year of education; and 17 pre-service teachers in their fourth year of education. It should be mentioned that the pre-service teachers did not attend a geometry course during their second year of education; hence, second-year pre-service teachers were not included in the present study. Moreover, the research population included 24 in-service mathematics teachers studying toward their M.Ed. in elementary school mathematics education, as well as 25 participants with a BA or MA degree in another discipline, who were making a career change to mathematics teaching at elementary school and junior high school.

Research Design

Research tools

The research tools included an attainments questionnaire with multiplechoice questions designed to determine the respondent's attainments and thinking level according to van Hiele's theory. The questionnaire comprised 45 close-ended questions. Five optional answers were given to each question and the respondents had to choose the correct one. The 45 questions consisted of 15 questions about three-dimensional geometric figures and 30 questions related to various topics of plane geometry (15 dealt with triangles and quadrilaterals and the other 15 related to circles). Every five questions in a group of 15 represented a separate thinking level on a higher hierarchical order.

The questions were presented in the following order: five questions at the first level dealing with triangles and quadrilaterals, then five questions at the first level associated with circles and five questions about three-dimensional geometric figures. Afterwards there were five additional questions about each of the topics representing the second thinking level and finally a group of 15 questions about the three topics representing the third thinking level. The time allocated to answering the questionnaire was 45 minutes. All the questions were based on previous questionnaires developed by Patkin (2010) and Patkin and Levenberg (2004, 2010), which have been processed and validated.

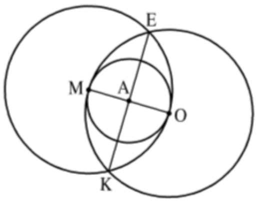
The test reliability was 0.84. Validation of the content was determined by a logical-scientific analysis. The items were sent to five judges, all of them researchers in the field of mathematics education. These judges were requested to categorize the items into the different levels according to van-Hiele's theory, indicating any items which were irrelevant or which did not comply with the appropriate criteria. Moreover, they were asked to point out unclear formulation which might have resulted in misunderstanding of the item. Items with concurred assertions of less than four judges were removed from the questionnaire.

Examples of Questions

The questions at the first level include questions of identification and distinction. As mentioned above, at this initial level, learners can identify geometric shapes and discern between them according to a drawing. Each of the concepts or the shapes is perceived as a whole, as it is seen. At this level, learners have not mastered the properties of those shapes.

The following is an example of a question at the first level dealing with the topic of a circle.

Below is a drawing of three circles: O, M, and A.



Which of the following claims is correct?

a. In circle A — MA is the diameter;
b. In circle O — KE is the diameter;
c. In circle A — MO is the diameter
d. In circle O — MO is the diameter;
e. In circle M — KE is the diameter.

In order to answer the question, learners should focus on each of the circles separately and identify whether the segment indicated in it is indeed a diameter.

The questions at the second level relate to geometric properties. At this level, learners are supposed to identify a certain shape, be aware that it has several properties, and check the existence or non-existence of the properties with regard to the shape.

An example of a question at the second level on the topic of three-dimensional geometric figures is as follows:

Which of the following claims is true regarding every prism?

a. A prism envelop is entirely built of triangles;
b. A prism has two parallel bases;
c. The prism bases are rectangular;

d. The prism envelope consists of regular polyhedron;
e. Four sides meet at every prism vortex.

The third level includes questions relating to relationships of connection between the various shapes and solids and to drawing conclusions. At this level, learners can manipulate the properties and derive information about the properties of the shapes based on other properties presented to them.

Here is an example of a question at the third level on the topic of three-dimensional geometric figures:

Which of the following claims is true?

a. If the geometric figure has 8 vertices, it is necessarily a box;
b. If the geometric figure has 8 vertices, it is necessarily a cube;
c. If the geometric figure has 8 vertices, it is necessarily a pyramid;
d. If the geometric figure has 8 vertices, it is necessarily a regular geometric figure;
e. All the above claims are incorrect.

Research Procedure

At the beginning of the academic year, the questionnaire was administered to each of the different groups of participants during various geometry courses which they attended according to their year of education (first, third, and fourth year) and to the different pathways in which they studied (M.Ed. or career change). In the first course session, the pre-service teachers were requested to respond anonymously to the questionnaire. They were explicitly told that the aim of the questionnaire was to map their knowledge in order to improve the teaching method of lecturers in those courses. The participants were told that no score would be given for responding to the questionnaire and that they did not have to write their name on it.

Analysis Tools

The participants' attainments in the various thinking levels were investigated according to the number of their correct answers, namely calculating the mean raw scores (as a percentage). Mastery of thinking levels relating to the various topics was determined according to the weighted scores. It is important to remember that mastering a certain level requires from learners to be versed in each of its previous levels.

The weighted scores were set according to a scale suggested by Usiskin (1982) and Patkin (1990), whereby at least 4 correct answers out of five answers on each level signified meeting the level criterion and accredited participants with one point. This reduced the chances of guessing and/or correct answers based on wrong thinking.

1. *Way of scoring the answers*
   - Giving at least four out of five correct answers at the first level awards one point.
   - Giving at least four out of five correct answers at the second level awards two points.
   - Giving at least four out of five correct answers at the third level awards four points.
2. *Way of weighing the scores* — A weighted score is comprised in the following way: complying with the criterion at the first level + complying with the criterion at the second level + complying with the criterion at the third level and so on. If a is the variable representing compliance with a criterion at any level, a can get the values 0 (failing to comply with the criterion) or 1 (complying with the criterion because the learner has given four out of five correct answers). In that case the weighted score can be represented in the following way: $a \cdot 1 + a \cdot 2 + a \cdot 4$ = weighted score.

Hence, the score range which relates to mastery of three out of first four levels of thinking in geometry is between 0–7 for each topic (triangles and quadrilaterals, the first topic; circles, the second topic; and solids, the third topic) (Patkin, 1990, 2010).

In light of the above, according to van Hiele's theory, no learner can be at level X before having mastered level X−1. That is, learners must be versed in all previous levels of thinking; otherwise they are referred to as "inconsistent." The weighted scores facilitate identification of learners' levels of thinking as follows: learners who have not mastered the first level of thinking will get the score of 0. Learners who have mastered the first level of thinking will get the score of 1. Learners versed in the second level of thinking will get the score of 3 ($1 \cdot 1 + 1 \cdot 2$) and learners versed in the third level of thinking will get the score of 7 ($1 \cdot 1 + 1 \cdot 2 + 1 \cdot 4$). The other scores represent "inconsistent" learners in the examined topic.

Findings

The first research question focused on the difference in geometric thinking levels of pre-service and in-service mathematics teachers at different points during their education in geometry in general and in each of the three topics (triangles and quadrilaterals, circles, and three-dimensional geometric figures) in particular.

Table 1 illustrates the means in percent (*M*) and standard deviation (*SD*) of the participants' raw scores in each of the five groups of the present study. As mentioned above, every correct answer accredited the participant with 1 point. Accordingly, every participant received a score of 0–45 (the questionnaire included 45 questions).

It is important to point out that the scores were converted from a scale of 0–45 to a scale of 0–100. Thus the mean scores (in percent) and the standard deviation of all the participants in each group was calculated.

Table 1: Mean Raw Scores (in Percent) and Standard Deviation in van Hiele's Questionnaire

	M	*SD*
First year (*n* = 46)	57	13.87
Third year (*n* = 30)	74	12.42
Fourth year (*n* = 17)	74	7.57
M.Ed. (*n* = 24)	74	10.52
Career change (*n* = 25)	67	13.08

Analysis of findings was performed by presenting frequencies and standard deviations since the number of participants is not big enough for checking significance.

Table 1 shows that the participants' mean raw scores in each of the different groups did not exceed 74. Furthermore, the mean scores in the first year was lower vis-à-vis the mean scores of the other participating groups. The findings indicate that the mean raw scores of third- and fourth-year pre-service teachers as well as in-service teachers studying toward an M.Ed. degree were identical. As to the mean score of the career-changing academics, the table illustrates that it was higher than that of the first-year pre-service teachers but lower than that of pre-service teachers in more advanced years and of in-service teachers studying toward an M.Ed. degree.

Table 2 presents the mean scores and standard deviation of pre-service teachers' raw scores for each of the investigated topics: triangle and quadrilaterals, circles, and three-dimensional geometric figures. Each topic included 15 questions and the mean score (in percent) and standard deviation of all the participants in that group were calculated.

Table 2: Mean of Raw Scores (in Percent) and Standard Deviation Classified According to Three Topics

	Triangles and quadrilaterals		Circles		Three-dimensional geometric figures	
	M	*SD*	*M*	*SD*	*M*	*SD*
First year (*n* = 46)	68	2.70	56	2.74	46	2.04
Third year (*n* = 30)	83	2.17	61	2.45	80	2.60
Fourth year (*n* = 17)	84	1.30	68	2.21	69	2.08
M.Ed. (*n* = 24)	83	1.84	68	1.72	71	3.37
Career change (*n* = 25)	79	2.39	67	2.60	56	2.21

Table 2 indicates that in each of the participating groups, the attainments relating to triangles and quadrilaterals were higher than those for the other two topics (circles and three-dimensional geometric figures). The findings shows that the attainments of the first-year pre-service teachers were lower (a mean score of 68%) than the attainments of the participants in the other four groups (ranging from 79–84%). Concerning the topic of circles, the attainments of the first-year pre-service teachers were also the lowest (56%). However, the attainments of the participants in the other four groups were particularly low and ranged from 61–68% on average. Regarding three-dimensional geometric figures, the first-year pre-service teachers had particularly low scores (46%). The score range of the other participants in the questions about three-dimensional geometric figures (56–80%) was wider than the ranges in the other two topics. The third year pre-service teachers scored relatively high (80%) whereas the fourth-year pre-service teachers and teachers studying toward their M.Ed. scored on average 69% and 71% respectively. Moreover, the career-changing academics did not attain high scores, receiving only 56% on average.

We will now examine the differences in thinking levels of the various groups taking part in the present study for each of the three investigated topics.

Table 3 presents the mean scores and standard deviations of the participants' raw scores for each of the three topics included in the questionnaire, according to the different thinking levels. For each correct answer, the participants were accredited with 1 point. Hence, every participant got a score between 0–5 (each level had 5 questions about each of the different topics) and the mean score and standard deviation of all the participants in that group were calculated. Figure 1 presents these findings as a bar chart.

Table 3 and Figure 1 illustrate that on the topic of triangles and quadrilaterals, participants in all the five groups gave at least 3 correct answers on average, for each of the thinking levels. Conversely, with regard to the topic of circles, an average of the correct answers at level 3 was lower than at the first and second level for this topic. A similar picture is obtained also for the topic of three-dimensional geometric figures.

The second research question explored at what thinking levels do pre-service and in-service mathematics teachers master the three topics examined in the present study.

Table 3: Mean Raw Scores for Each Level (Range 0–5) and Standard Deviation for Each of the Topics

	Triangles and quadrilaterals						Circles						Three-dimensional geometric figures					
	Level 1		Level 2		Level 3		Level 1		Level 2		Level 3		Level 1		Level 2		Level 3	
	M	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*
First year (n = 46)	3.33	0.91	3.83	1.24	3.04	1.38	3.72	0.90	2.83	1.39	1.89	1.27	2.78	0.88	2.39	0.77	1.67	1.00
Third year (n = 30)	4.13	0.72	4.27	1.12	4.03	0.98	3.83	0.97	3.10	1.30	2.17	1.04	4.30	0.90	3.87	1.06	3.80	1.33
Fourth year (n = 17)	4.24	0.55	4.35	1.03	3.94	0.94	4.29	0.67	3.59	1.37	2.35	1.08	4.06	0.87	3.00	1.19	3.29	1.07
M.Ed. (n = 24)	4.20	0.80	4.56	0.80	3.76	1.24	4.20	0.75	3.52	1.02	2.48	1.06	4.08	1.23	3.32	1.22	3.20	1.55
Career change (n = 25)	4.13	0.67	4.13	1.01	3.58	1.15	4.13	0.94	3.71	1.02	2.29	1.43	3.42	0.81	2.96	0.73	2.00	1.25

Figure 1: Mean of Raw Scores (Range 0–5) for Each of the Topics

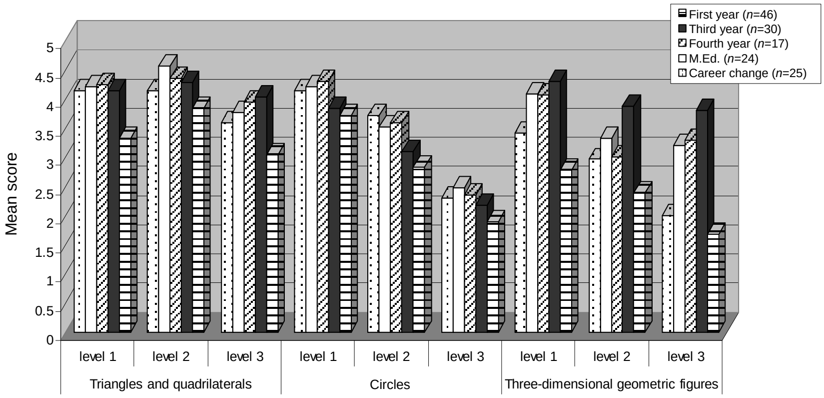


Table 4 presents the percentage of participants responding correctly to at least 80% of the questions on each topic and at each level. One should note that at each of the levels, there were five questions which focused on a certain topic (five questions on triangles and quadrilaterals, five on circles, and five on three-dimensional geometric figures). Table 4 shows the percentage of participants who responded correctly to at least four questions about each of the topics and at each of the levels.

The findings indicate that on the topic of triangles and quadrilaterals, the pre-service teachers in their first year and the in-service teachers studying for an M.Ed. degree gave a higher percentage of correct answers to at least four out of the five questions at the second level than the percentage of correct answers at the first level. That is, these participants knew to answer correctly questions associated with properties of triangles and quadrilaterals better than identifying and naming them. This was not the case for the findings of the other two geometric topics (circles and three-dimensional geometric figures). It was generally found that the percentage of participants who gave at least four correct answers decreased as the level of difficulty of the questions increased (according to thinking levels).

Table 5 and Figure 2 display the mastery of geometry thinking levels according to van Hiele's theory for each of the five groups. As

Table 4: Percentage of Participants Who Responded Correctly to No Less Than 4 Questions Out of a Group of 5 Questions on the Same Topic and Same Level

	Triangles and quadrilaterals						Circles						Three-dimensional geometric figures					
	Level 1		Level 2		Level 3		Level 1		Level 2		Level 3		Level 1		Level 2		Level 3	
	M	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*	*M*	*SD*
First year (n = 46)	39	0.49	65	0.48	44	0.50	59	0.49	37	0.48	9	0.28	15	0.36	7	0.25	2	0.15
Third year (n = 30)	87	0.34	87	0.34	67	0.47	57	0.50	43	0.50	7	0.25	83	0.37	67	0.47	63	0.48
Fourth year (n = 17)	94	0.24	82	0.32	76	0.42	88	0.32	59	0.49	12	0.32	76	0.42	29	0.46	47	0.50
M.Ed. (n = 24)	84	0.37	92	0.27	76	0.43	80	0.40	56	0.50	12	0.32	72	0.45	48	0.50	56	0.50
Career change (n = 25)	83	0.37	79	0.41	42	0.49	75	0.43	63	0.48	25	0.43	54	0.50	17	0.37	13	0.33

mentioned above, when calculating the weighted score, in order to determine the participants' thinking level, participants who have not mastered the first level get a score of 0 and those who have mastered the first level get a score of 1. Participants versed in the second level should have mastered the first and second levels and, therefore, would get a score of 3. Participants versed in the third level get a score of 7 (mastery of all the previous levels as well as the present one). The other scores represent the "inconsistent" participants.

Table 5 and Figure 2 indicate that most of the pre-service mathematics teachers at different points of their studies have mastered the topic of geometric shapes in a plane (i.e., triangles and quadrilaterals). Moreover, they are at the third level in accordance with van Hiele's theory. The first-year pre-service teachers are less versed and a considerable number of them have mastered only the second level. More than one-third of the first-year pre-service teachers (35%) were found to be inconsistent in their mastery. That is, they knew the properties of those geometric figures but encountered difficulties in identifying them (first level) and consequently were considered as inconsistent. As to the topic of circles, most of the participants have mastered only the second level and that applies to all the five groups. However, regarding three-dimensional geometric figures, most of the participants were only at the first or even zero level. They are not versed in the higher levels of properties knowledge (second level) and the informal deduction (third level).

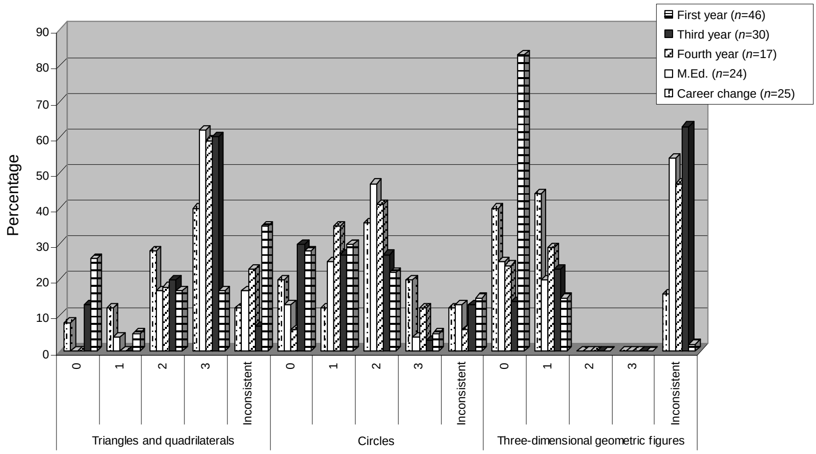
To sum up, the findings illustrate that the level of thinking of the pre-service teachers at the beginning of their first academic year and of the career-changing academics was lower than the participants learning in their third or fourth year of education as well as the in-service teachers. Moreover, analysis of the findings shows that on the topic of triangles and quadrilaterals, the participants in this study demonstrated mastery at the third level of thinking. Conversely, on the topic of circles, most of the participants were versed only in the second level and on the topic of three-dimensional geometric figures, they mastered the first level of thinking at the most.

Table 5: Mastery (%) of Thinking Levels According to van Hiele's Theory Based on the Weighted Scores

	Triangles and quadrilaterals					Circles					Three-dimensional geometric figures				
	Level					Level					Level				
	0	1	2	3	Inconsistent*	0	1	2	3	Inconsistent*	0	1	2	3	Inconsistent*
Weighted score	0	1	3	7	other	0	1	3	7	other	0	1	3	7	other
First year (*n* = 46)	26	5	17	17	35	28	30	22	5	15	83	15	0	0	2
Third year (*n* = 30)	13	0	20	60	7	30	27	27	3	13	14	23	0	0	63
Fourth year (*n* = 17)	0	0	18	59	23	6	35	41	12	6	24	29	0	0	47
M.Ed. (*n* = 24)	0	4	17	62	17	13	25	47	4	13	25	20	0	0	54
Career change (*n* = 25)	8	12	28	40	12	20	12	36	20	12	40	44	0	0	16

* "Inconsistent" participants have not mastered the cognitive level according to the definition.

Figure 2: Mastery (%) of Thinking Levels According to van Hiele's Theory Based on the Weighted Scores



Discussion and Conclusions

The present study investigated the difference in thinking levels of pre-service and in-service mathematics teachers in general with respect to three main topics (triangles and quadrilaterals, circles, and three-dimensional geometric figures) in geometry studied in Israel, according to the mathematics curriculum at elementary school. Furthermore, the study explored the thinking level of pre-service mathematics teachers in the first, third, and fourth year of education, of teachers studying toward an M.Ed. degree, and of academics making a career change to mathematics teachers with regard to the three different topics.

The questions given to the participants related to identification of two- and three-dimensional geometric figures (first level according to van Hiele), properties of those figures (second level according to van Hiele), and the ability to draw conclusions in an informal way (third level according to van Hiele). Studies of geometric capability of pre-service teachers indicate that most participants have usually mastered the first two levels but only a few are versed in the third level (Gutiérrez, Jaime, & Fortuny, 1991; Halat & Şahin, 2008; Patkin & Sarfaty, 2012).

Findings of the present study corroborated the conclusions of these studies. All the research participants have a full matriculation certificate, including mathematics and some of them learnt mathematics at an advanced level at high school. Nevertheless, in spite of their mathematics studies, including geometry, throughout their twelve years at elementary and high school, they were versed only in the first thinking level regarding circles and three-dimensional geometric figures. They failed to master the two higher thinking levels (second and third levels), particularly on the topic of three-dimensional geometric figures. It is noteworthy that with regard to triangles, they were versed in the third thinking level. The fact that in Israel, the topic of triangles and quadrilaterals is taught already from the 3rd until the 12th grade can account for that. The topic of circles was taught only in the 6th grade, mainly with reference to naming and identification of the figures (at the first level according to van Hiele's theory). Pupils who studied mathematics at an advanced level had learnt it once throughout high school at a formal level, without paying attention to previous knowledge, developing competences and basic skills, and so on. The topic of three-dimensional geometric figures was taught even less in the past. It was supposed to be learnt in the 6th grade. However, teachers without enough time for teaching did not teach this topic because it was not legally enforced. In high school, once more only pupils in advanced classes learnt it in higher grades, over a short period of time, at the technical level of memorization and procedures and mainly from the aspect of vectors.

Due to the dissatisfaction with pupils' achievements in international mathematics tests such as TIMSS and PISA as well as in national mathematics tests (Gonzales et al., 2009) and in order to improve pupils' achievement, the Israeli education authority decided to change the mathematics curriculum for elementary school and to put more focus on geometry — both plane geometry and solid geometry (Ministry of Education, 2006). It also decided to change the junior high school curriculum, a change which is being implemented these days, particularly in the teaching of solids and circles (Ministry of Education, 2013). The topic of solids was taught in the past from the 6th grade and, later, only in the higher grades of high school within the framework of enhanced mathematics studies. This topic is now studied in a spiral and gradual way from the 2nd grade until the end of junior high school. The objective was that pupils graduating the 9th grade will master at least the first thinking levels of this topic. Moreover, teaching the topic of circles has been transformed although, in fact, the age group in which it was learnt for the first time was not changed (as mentioned above, it was done in the 6th grade). In the past, after learning the topic of circles in the 6th grade, the topic was not learnt in junior high school (7th to 9th grades) and it was taught once more only in the 10th grade. Now it is learnt in all these grades, aiming to deepen and preserve what has been learnt at elementary school, develop geometric thinking by integrating visualization aids (cutting, folding, models), and learn to solve every-day literacy problems. Thus, in the 10th grade, pupils who have chosen to specialize in mathematics within the framework of enhanced mathematics studies will make a progress and master the topic also at higher levels of thinking.

Teachers play an essential role in the promotion of pupils' mathematics knowledge and the implementation of a new curriculum. Teachers with appropriate mathematics knowledge will be able to offer their pupils opportunities of a meaningful learning (American Federation of Teachers, 2008).

With reference to the different points of time in the participants' education, the findings show that the thinking level of first-year pre-service teachers and that of the career-changing academics is lower than the thinking level of third- or fourth-year pre-service teachers and that of in-service teachers. This might be explained by the fact that geometry teaching in teacher education college combines studies of required geometric contents while emphasizing the learner's necessary competences and skills, according to van Hiele. Experiencing in and focusing on didactic aspects, aiming to facilitate the learner's progress in mastering thinking levels, led third- and fourth-year pre-service teachers to master higher thinking levels than career-changing academics or first-year pre-service teachers. These findings are in line with van Hiele's theory, underscoring that the transition from one thinking level to another mainly depends on teaching or, more precisely, on teaching quality (Geddes et al., 1982; Halat & Şahin, 2008; van Hiele, 1999). Teaching which consists of memorization and repetition of the same contents without developing required skills and competences does not enhance the learner's thinking levels. The findings for the third- and fourth-year pre-service teachers and the in-service teachers were similar. This is because in-service teachers were studying in teacher education colleges, and hence were exposed to geometry in theory and to required skills. These improved their mastery of the higher thinking levels in geometry.

Findings of the present study point out two aspects. The first aspect is that the factor which might improve geometry mastery level of pre-service and in-service teachers is their studies at the teacher education college. This is grounded in the fact that the mastery level of pre-service and in-service teachers in their advanced stages of training is higher than that of those who have just started their education process.

The second aspect relates to the mathematics education given in Israeli colleges. It seems that the subjects, the teaching methods and the activities offered to pre-service teachers should be examined in a comprehensive way during the teaching of geometry and its different branches in teacher education colleges. All those engaged in teaching this subject should be better versed in it so that they have a sufficient knowledge base for teaching geometry to their elementary school pupils. As already mentioned, the type of activities given to learners is meaningful and compliance between their comprehension level and assignments level is crucial if we want to have meaningful learning (Crowley, 1987). Nevertheless, not enough is being done in some of the learning topics, particularly as far as circles and three-dimensional geometric figures are concerned. Consequently, it is necessary to add learning hours and courses which will give practice to teachers for mastering the high thinking levels of these topics so that they can teach the topics themselves. The similar findings relating to in-service teachers and third- and fourth-year pre-service teachers reinforce the need for expanding the knowledge of pre-service and in-service mathematics teachers at each stage of their education and professional development in the different topics of geometry, mainly circles and three-dimensional geometric figures. In these areas, all the participants demonstrated lack of mastery and weak attainments.

As a recommendation and summary, it is important to reiterate that this article provides only a general picture of the various levels of thinking of in-service and pre-service teachers regarding different topics in geometry. Consequently, continuous and additional studies of the difficulties and misconceptions embodied in each of the topics, whereby mastery of the thinking levels is low, is recommended. As a result, teacher-educators should give priority to teaching topics which are less familiar to pre-service teachers (focusing less on familiar topics). Hence, in-service and pre-service teachers would master both topics — circles and three-dimensional geometric figures — at the highest levels of thinking.

Moreover, further study can be conducted with a larger research population. With this, it is possible to perform significance tests for the purpose of generalization and conclusion drawing. It is also recommended to explore whether in-service training courses and intervention programs can promote teachers and learners' levels of thinking with regard to the various topics.

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