Construction of fractional spline wavelet bases

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ABSTRACT

We extend Schoenberg's B-splines to all fractional degrees $\alpha > -\frac{1}{2}$. These splines are constructed using linear combinations of the integer shifts of the power functions x_+^α (one-sided) or $|x|_*^\alpha$ (symmetric); in each case, they are α-Hölder continuous for $\alpha > 0$. They satisfy most of the properties of the traditional B-splines; in particular, the Riesz basis condition and the two-scale relation, which makes them suitable for the construction of new families of wavelet bases. What is especially interesting from a wavelet perspective is that the fractional B-splines have a fractional order o
f approximation $(\alpha+1)$, while they reproduce the polynomials of degree $\lceil \alpha \rceil$. We show how they yield continuous-order generalizations of the orthogonal Battle-Lemarie wavelets and of the semi-orthogonal B-spline wavelets. As α increases, these latter wavelets tend to be optimally localized in time and frequency in the sense specified by the uncertainty principle. The corresponding analysis wavelets also behave like fractional differentiators; they may therefore be used to whiten fractional Brownian motion processes.

Keywords: splines, wavelet design, fractional derivatives, approximation order, Strang-Fix theory.

1. INTRODUCTION

Splines have had a significant impact on the early development of the theory of the wavelet transform [18, 5, 8, 10, 12]. In fact, they constitute a case apart for they yield the only wavelets that have an explicit analytical form. All other wavelet bases are defined indirectly through an infinite recursion (or an infinite product in Fourier domain). To date, four sub-families of spline wavelets have been characterized explicitly: the orthogonal Battle-Lemarié wavelets [5, 8], the semi-orthogonal spline wavelets [6, 21, 22], the biorthogonal splines [7], and the shift-orthogonal spline wavelets [24]. The first two types span the same spline multiresolution subspaces and are based on an orthogonal projection; the more general semi-orthogonal splines are orthogonal with respect to dilation but not necessarily with respect to shifts. The two remaining subtypes involve two multiresolution analyses instead of one and implement an oblique projection [1]. The more constrained ones are the shift-orthogonal wavelets which are orthogonal with respect to shifts but not with respect to dilation. One notable property is that these splines — irrespective of their type — appear to have the best approximation properties among all known wavelet families: they yield the smallest asymptotic (scale-truncated) approximation error for a given order L [19, 20].

In this paper, we propose to generalize these spline constructions to obtain new wavelet bases with a continuous order parameter. For this purpose, we first extend the construction of B-splines to fractional degrees. These functions will be indexed by a continuous parameter $\alpha > -\frac{1}{2}$ which represents the Hölder exponent of the fractional spline. They interpolate the conventional B-splines which correspond to the special case where α is integer. We will show that these new fractional splines share virtually all the properties of the polynomial splines with the exception of compact support when α is non-integer. Most importantly, they satisfy a two-scale relation which is the key to construction of wavelet bases. This will allow us to construct wavelet bases parametrized by the continuously-varying regularity parameter α. We will see that these new spline wavelets have some remarkable properties. In particular, they have a fractional order of approximation, a property that has not been encountered before in wavelet theory. They also behave like fractional differentiation operators, while standard wavelets only give integer orders of differentiation.

The paper is organized as follows. In Section 2, we introduce the two versions (causal and symmetric) of fractional B-splines. In Section 3, we look at their most important properties (fractional differentiation rules, Riesz bounds, two-scale relation, and fractional orders of approximation); for more complete treatment, we refer to [23]. We then use these results in Section 4 to produce spline wavelets bases with a continuous order parameter: these include generalizations of the Battle-Lemarié wavelets, which are orthogonal, and the B-spline wavelets which are of interest because of their near optimal time-frequency localization properties.

1.1 Notations and definitions

A. Gamma function and generalized binomials

Euler's gamma function, which generalizes the factorial, is defined as

$$\Gamma(u+1)=\int_0^{+\infty} x^u e^{-x}\,dx; \tag{1}$$

in particular, $\Gamma(n+1)=n!$. This suggest the following generalization of the Binomial coefficients

$$\binom{u}{v}:=\frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)} \tag{2}$$

In particular, this definition implies that $\binom{u}{k}=0$ when k is a strictly negative integer. Moreover, for $u\geq 0$, we have the well-known binomial theorem

$$(1+z)^u=\sum_{k=0}^{\infty}\binom{u}{k}z^k$$

When $u=n$ (integer), $\binom{n}{k}$=0 for $k\geq n+1$ and one recovers the standard Binomial expansion.

B. Fractional derivatives

The differentiation operator can be extended to non-integer exponents rather simply in the Fourier domain:

$$D^\alpha f(x) \quad \stackrel{Fourier}{\longleftrightarrow} \quad (j\omega)^\alpha \hat{f}(\omega) \tag{3}$$

where $\hat{f}(\omega)=\int f(x)e^{-j\omega x}dx$ denotes the Fourier transform of $f(x)$ and where $z^\alpha=|z|^\alpha e^{j\alpha\arg(z)}$ with $j=\sqrt{-1}$ and $\arg(z)\in[-\pi,\pi[$. This is equivalent to Liouville's definition of fractional derivative operator [9].

C. Fractional finite differences

The forward fractional finite difference operator of order α is defined as follows

$$\Delta_+^\alpha f(x):=\sum_{k=0}^{+\infty}(-1)^k\binom{\alpha}{k}f(x-k) \tag{4}$$

It is a *convolution* operator which has a more straightforward interpretation in the Fourier domain

$$\hat{\Delta}_+^\alpha(\omega) = (1 - e^{-j\omega})^\alpha = \sum_{k=0}^{+\infty} (-1)^k \binom{\alpha}{k} e^{-j\omega k}. \qquad (5)$$

This definition insures that the operator coincides with the conventional one when α is an integer.

D. Power functions

The natural building blocks for the fractional splines are the one-sided power functions x_+^α, which have precisely one singularity of order α (Hölder exponent) at the origin:

$$x_+^\alpha = \begin{cases} x^\alpha & x \geq 0 \\ 0, & \text{otherwise} \end{cases}. \qquad (6)$$

Their Fourier transform is $\Gamma(\alpha+1)/(j\omega)^{\alpha+1}$.

For symmetry and notational purposes, we introduce the functions $|x|_*^\alpha$ whose Fourier transform is $\Gamma(\alpha+1)/|\omega|^{\alpha+1}$. For α non-even, they are power functions as well; otherwise, they have an additional logarithmic factor:

$$|x|_*^\alpha = \begin{cases} \dfrac{|x|^\alpha}{-2\sin(\frac{\pi}{2}\alpha)}, & \alpha \text{ not even} \\ \dfrac{x^{2n}\log x}{(-1)^{1+n}\pi}, & \alpha = 2n \text{ (even)} \end{cases} \qquad (7)$$

2. FRACTIONAL B-SPLINES

In this section, we summarize the main properties of the fractional B-splines. For more details, we refer to [23].

2.1 Causal fractional B-splines

By analogy with the classical B-splines, we define the fractional causal B-splines by taking the $(\alpha+1)$th fractional difference of the one-sided power function

$$\beta_+^\alpha(x) := \frac{\Delta_+^{\alpha+1} x_+^\alpha}{\Gamma(\alpha+1)} = \frac{1}{\Gamma(\alpha+1)} \sum_{k=0}^{+\infty} (-1)^k \binom{\alpha+1}{k} (x-k)_+^\alpha. \qquad (8)$$

These functions are in L_2 for $\alpha > -\frac{1}{2}$; they decay at least like $|x|^{-(\alpha+2)}$ (cf. [23], Theorem 3.1). Some examples of fractional B-splines are shown in Fig. 1. While they seem to be decaying reasonably rapidly, they are not compactly supported unless α is an integer, in which case we recover the classical B-splines [14, 15]. In general, they do not have an axis of symmetry either.

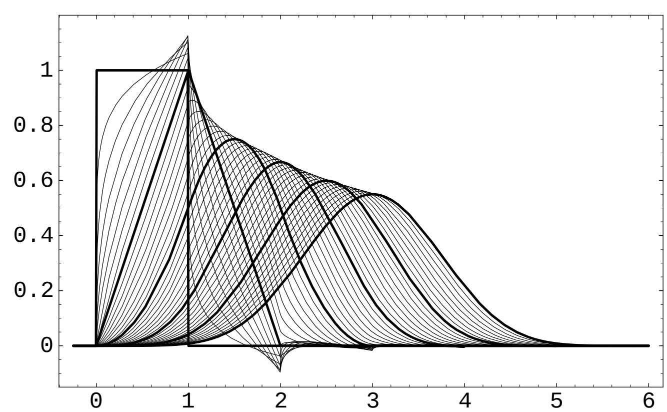


Fig. 1: The fractional B-splines with α≥0. These functions interpolate the conventional B-splines which are represented using a thicker line.

The fractional causal B-splines satisfy the convolution property

$$\beta_+^{\alpha_1} * \beta_+^{\alpha_2} = \beta_+^{\alpha_1+\alpha_2+1}. \tag{9}$$

2.3 Symmetric fractional B-splines

The symmetric versions of the fractional B-splines of degree α are defined as

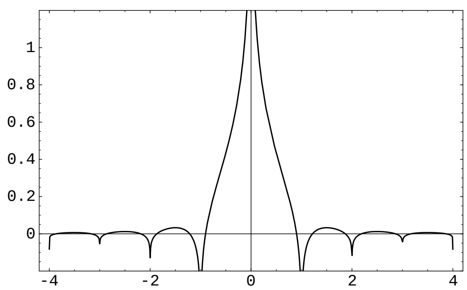
$$\beta_*^{\alpha}(x) := \beta_+^{\frac{\alpha-1}{2}} * \beta_-^{\frac{\alpha-1}{2}}. \tag{10}$$

We are able to derive an explicit time domain formula (c.f. [23]).

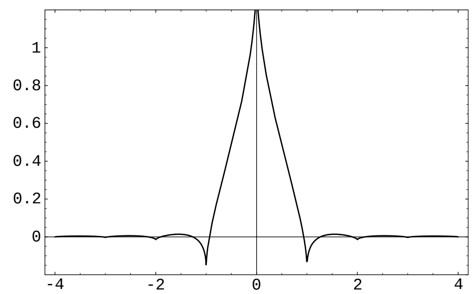
Theorem. *The centered fractional B-splines of degree α are given by*

$$\beta_*^{\alpha}(x) := \frac{\Delta_*^{\alpha+1}|x|_*^{\alpha}}{\Gamma(\alpha+1)} = \frac{1}{\Gamma(\alpha+1)} \sum_{k \in Z} (-1)^k \left| \begin{matrix} \alpha+1 \\ k \end{matrix} \right| \, |x-k|_*^{\alpha} \tag{11}$$

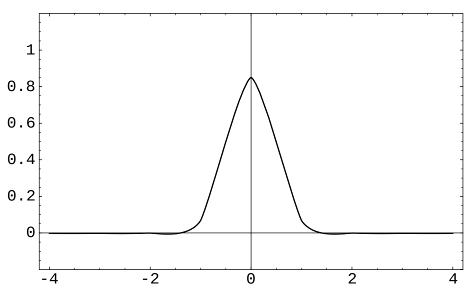
where $|x|_*^{\alpha}$ is given by (7) and where $\Delta_*^{\alpha} \xleftrightarrow{Fourier} \left|1-e^{-j\omega}\right|^{\alpha}$ is a symmetrized version of the fractional difference operator.



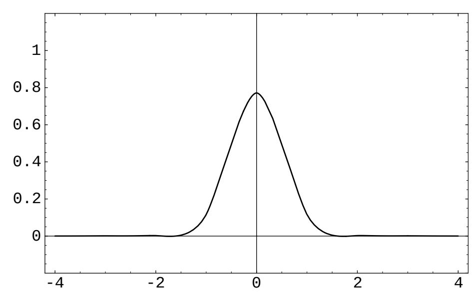
(a) $\alpha = 0$



(b) $\alpha = 1/2$



(c) $\alpha = 3/2$



(d) $\alpha = 2$

Fig. 2: Examples of symmetric fractional B-splines of increasing regularity.

The above expansion involves modified binomial coefficients, which are defined through the generating function

$$|1+z|^{\alpha} := \sum_{k \in Z} \left| \begin{matrix} r \\ k \end{matrix} \right| z^k, \quad \text{for } z = e^{-j\omega}$$

which is convergent only on the unit circle. In fact, these coefficients are a re-centered version of generalized binomials, for it can be shown that

$$\left| \begin{matrix} r \\ k \end{matrix} \right| = \binom{r}{k+\frac{r}{2}}. \tag{12}$$

Some examples of these fractional centered B-splines are shown in Fig. 2. Similar to their causal counterparts, they are α-Hölder continuous with knots at the integers; they are not compactly supported either unless *n* is odd. When α is not odd, they decay like $|x|^{-(\alpha+2)}$ and their asymptotic form is available (cf. [23], Theorem 3.1). The most notable difference is that our centered B-splines are constructed using the integer shifts of $|x|^\alpha$ rather than $(x)_+^\alpha$. Also note that they do only coincide with the standard centered B-splines when the degree is *odd* ; this is because of the absolute value, which creates a discontinuity in the Fourier domain when α is even. For comparison, Schoenberg's centered B-spline of even degree have knots at the half integers; they therefore span different spaces.

3. FRACTIONAL B-SPLINES PROPERTIES

We use the generic notation $\beta^\alpha(x)$ to specify any one of the fractional B-splines ($\beta_+^\alpha(x)$, $\beta_-^\alpha(x)=\beta_+^\alpha(-x)$, or $\beta_*^\alpha(x)$).

3.1 Fourier transform

The Fourier transform of the fractional B-splines are given by

$$\hat{\beta}_+^\alpha(\omega)=\left(\frac{1-e^{-j\omega}}{j\omega}\right)^{\alpha+1} \tag{13}$$

$$\hat{\beta}_*^\alpha(\omega)=\left|\frac{\sin(\omega/2)}{\omega/2}\right|^{\alpha+1} \tag{14}$$

These equations are obviously compatible with the convolution property (8).

3.2 Fractional derivatives and regularity

One of the primary reasons for the success of polynomial splines in applications is that they can be differentiated very simply by taking finite differences [13]. This property generalizes nicely to the fractional case:

$$D^\gamma\beta_+^\alpha(x)=\Delta_+^\gamma\beta_+^{\alpha-\gamma}(x) \tag{15}$$

$$D_*^\gamma\beta_*^\alpha(x)=\Delta_*^\gamma\beta_*^{\alpha-\gamma}(x) \tag{16}$$

where we use the following definition of operators:

(*i*) differentiation:

$$D^\alpha \quad \xleftrightarrow{\;Fourier\;} \quad (j\omega)^\alpha$$

$$D_*^\alpha \quad \xleftrightarrow{\;Fourier\;} \quad |\omega|^\alpha \tag{17}$$

(*ii*) Finite differences:

$$\Delta_+^\alpha \quad \xleftrightarrow{\;Fourier\;} \quad (1-e^{-j\omega})^\alpha \tag{18}$$

$$\Delta_*^\alpha \quad \xleftrightarrow{\;Fourier\;} \quad \left|1-e^{-j\omega}\right|^\alpha \tag{19}$$

Since $\left|\hat{\beta}^\alpha(\omega)\right|=\hat{\beta}_*^\alpha(\omega)$ is bounded and decays like $|\omega|^{-\alpha-1}$, the fractional B-splines are in L_2 for $\alpha>-\frac{1}{2}$. More generally, one has

$$\beta^\alpha\in W_2^r \text{ with } r<\alpha+\frac{1}{2} \tag{20}$$

so that the critical Sobolev exponent of the fractional splines is $r_{\max}=\alpha+\frac{1}{2}$; that is, one half more than their Hölder exponent α.

3.3 Fractional spline spaces and Riesz bounds

The basic space of fractional splines of degree α with knots at the integers is defined as

$$S_+^\alpha = \left\{ s(x) = \sum_{k\in Z} c(k)\beta_+^\alpha(x-k) : c \in l_2 \right\}. \tag{21}$$

More generally, we may consider the spline subspaces S^α generated by $\beta^\alpha(x)$.

Proposition: *For* $\alpha > -\frac{1}{2}$, *the fractional B-spline* $\beta^\alpha(x)$ *generates a Riesz basis of* S^α. *Specifically, one has the following* l_2-L_2 *norm equivalence*

$$\forall c \in l_2, \quad A_\alpha \|c\|_{l_2}^2 \le \left\| \sum_{k\in Z} c(k)\beta^\alpha(x-k) \right\|_{L_2} \le B_\alpha \|c\|_{l_2}^2 \tag{22}$$

where A_α *and* B_α *are two constants such that*

$$A_\alpha \ge \left(\frac{2}{\pi}\right)^{2\alpha+2}$$

$$B_\alpha \le 1 + 2\zeta(2\alpha+2)\left(\frac{2}{\pi}\right)^{2\alpha+2}.$$

This result ensures that the B-spline representation (21) is stable and that the fractional spline spaces are well-defined (closed) subspaces of L_2. Starting from the B-splines, it is then easy, using the method described in [4], to generate other equivalent bases of these spaces with specific properties; for instance, orthogonality or interpolation.

3.4 Two-scale relation

The fractional B-splines have all the required multiresolution properties for the construction of wavelet bases. In particular, they satisfy the two-scale relation

$$\beta^\alpha(x/2) = \sum_{k\in Z} h^\alpha(k)\beta^\alpha(x-k). \tag{23}$$

The refinement filters are given by

$$h_+^\alpha(k) = \frac{1}{2^\alpha}\binom{\alpha+1}{k} \quad \longleftrightarrow \quad \hat{h}_+^\alpha(\omega) = 2\left(\frac{1+e^{-j\omega}}{2}\right)^{\alpha+1} \tag{24}$$

and

$$h_*^\alpha(k) = \frac{1}{2^\alpha}\left|\begin{matrix}\alpha+1\\ k\end{matrix}\right| \quad \longleftrightarrow \quad \hat{h}_*^\alpha(\omega) = 2\left|\frac{1+e^{-j\omega}}{2}\right|^{\alpha+1}. \tag{25}$$

Thus, our generalized binomial filter $h_+^\alpha(k) = h_-^\alpha(-k)$ is the natural extension of the binomial refinement filter for splines which plays such a central role in wavelet theory. Interestingly, for $-\frac{1}{2} < \alpha < 0$, the refinement filer $\hat{h}_+^\alpha(\omega)$ does not have the factor $(1+e^{-j\omega})$ which is usually believed to be necessary for the construction of unconditional wavelet bases of L_2. We will see that this is not a problem and that these low regularity splines can yield wavelets that are perfectly valid.

3.5 Fractional order of approximation

Another important ingredient for the construction of wavelet bases is the denseness of the representation in L_2. In other words, we need to make sure that the approximation error vanishes as the scale *a* goes to zero. We therefore consider the scale behavior of the orthogonal projection P_a which approximates a function on a fractional spline space at scale *a*. The main result is that the approximation error decays like $\|f - P_a f\| = O(a^{-(\alpha+1)})$ (cf. [23], Theorem 4.1):

Theorem. *The fractional splines have a fractional order of approximation* $\alpha+1$. *Specifically, the* L_2 *least squares approximation error is bounded by*

$$\forall f \in W_2^{\alpha+1},\ \|f - P_a f\|_{L_2} \le \left(\frac{1}{\pi^{\alpha+1}} \sqrt{2\zeta(\alpha+2) - \frac{1}{2}} \right) \|f^{(\alpha+1)}\|_{L_2} a^{\alpha+1} \tag{26}$$

and its asymptotic form is

$$\forall f \in W_2^{\alpha+1}, \quad \|f - P_a f\|_{L_2} = \left(\frac{\sqrt{2\zeta(2\alpha+2)}}{(2\pi)^{\alpha+1}} \right) \|f^{(\alpha+1)}\|_{L_2} a^{\alpha+1}, \text{ as } a \to 0. \tag{27}$$

where $\zeta(r) = \sum_{n\ge1} n^{-r}$ *is Riemann's zeta function.*

This is a remarkable result since fractional orders of approximation have never been encountered before in wavelet theory. Until now, the order of approximation has always been linked to the number of vanishing moments of the wavelet and restricted to being an integer [17].

3.5 Reproduction of polynomials

When we say that a function φ reproduces the polynomials of degree *n*, we mean that there exist some sequences $c_m(k)$ such that

$$\sum_{k\in Z} c_m(k)\varphi(x-k) = x^m, \qquad m = 0,\ldots,n. \tag{28}$$

Hence it follows that any polynomial of degree *n* is expressible as a linear combination of the integer shifts of φ. In the classical Strang-Fix theory of approximation [16], as well as in traditional wavelet theory [17, 20], there is an equivalence between the order L (which is an integer) and the reproduction of polynomials of degree $n = L-1$. This is not anymore the case here. In fact, the fractional B-splines of degree α (or order $\alpha+1$) reproduce polynomials of degree n where $n-1 < \alpha \le n$, which makes us jump to the next higher integer $\lceil \alpha \rceil$ when α is non-integer. Thus the non-integer part of α buys us one extra degree.

4. FRACTIONAL SPLINE WAVELETS

Let us now show explicitly how to use these B-splines to obtain new wavelet families with a continuously-varying order parameter. This task does not present any major conceptual difficulty; we can easily adapt any of the construction methods that are already available for polynomial splines. Here, we will concentrate on two extreme cases of wavelet bases: those that are orthogonal, and those that have the best localization.

4.1 Fractional Battle-Lemarié wavelets

These are obtained by orthogonalization of the B-splines. The key quantity is the fractional B-spline auto-correlation sequence:

$$a_\varphi(k) := \langle \beta^\alpha(x), \beta^\alpha(x-k) \rangle = \beta_*^{2\alpha+1}(k) \tag{29}$$

where the right hand side expression is a direct consequence of the convolution relations (7) and (9). Going to the Fourier domain, we get

$$A_\varphi(e^{j\omega}) = \sum_{k\in Z} \left| \hat{\beta}^\alpha(\omega + 2\pi k) \right|^2 = \sum_{k\in Z} \beta_*^{2\alpha+1}(k) e^{-j\omega k}. \tag{30}$$

Thus, to calculate $A_\varphi(e^{j\omega})$, we have essentially two strategies: either to perform an infinite summation in the Fourier domain, or to compute the B-spline samples $\beta_*^{2\alpha+1}(k)$ explicitly. In the standard polynomial case, the most efficient approach is clearly the second one because the B-splines are compactly supported [22]. Here, in the fractional case, there is no clear advantage of one method over the other. In our implementation, we have chosen the former for we have a relatively convenient Fourier domain expression for $\hat{\beta}^\alpha(\omega)$ (cf. (13) and (14)). Using the general approach in [2], we express the orthogonal scaling function as

$$\phi(x) = \sum_{k\in Z} (a_\varphi)^{-1/2} \beta^\alpha(x-k) \tag{31}$$

where $(a_\varphi)^{-1/2}$ denotes the convolution square-root inverse of a_φ. The corresponding two-scale relation is

$$\phi(x/2) = \sqrt{2}\sum_{k\in Z} h_o(k)\phi(x-k) \tag{32}$$

and the refinement filter is given by

$$H_o(e^{j\omega}) = \sqrt{\frac{A_\varphi(e^{j\omega})}{A_\varphi(e^{j2\omega})}}\sqrt{2}\left|\frac{1+e^{-j\omega}}{2}\right|^{\alpha+1}. \tag{33}$$

The corresponding orthogonal wavelet filter is obtained using Mallat's recipe: $G_o(z) = z\cdot H_o(-z^{-1})$ (cf. [11]).

4.2 Fractional B-spline wavelets

In this case, the scaling function is the fractional B-spline with its associated generalized binomial filter $H^\alpha(z)$ (cf. (24) and (25)). We are now searching for its corresponding semi-orthogonal wavelet which has the best possible localization in time. This can be achieved by direct extension of what has already been done for the polynomial splines [6, 21].

Here, the wavelet is expressed as

$$\psi^\alpha(x/2) = \sum_{k\in Z} g^\alpha(k)\beta^\alpha(x-k) \tag{34}$$

and the sequence $g^\alpha(k)$ needs to be determined such as to satisfy the semi-orthogonality[1] requirement: $\langle\beta^\alpha(\cdot),\psi^\alpha(\cdot-k)\rangle = 0$. The transfer function of the "shortest" (or most localized) wavelet filter is then given by

$$G^\alpha(z) = z\cdot A_\varphi(-z)\cdot H^\alpha(-z^{-1}). \tag{35}$$

Thus, we can write the explicit form of the fractional B-spline wavelets

$$\psi_+^\alpha(x/2) = \sum_{k\in Z}\frac{(-1)^k}{2^\alpha}\sum_{l\in Z}\binom{\alpha+1}{l}\beta_*^{2\alpha+1}(l+k-1)\beta_+^\alpha(x-k), \tag{36}$$

each of which yields a Riesz basis for L_2. The sequence of B-spline wavelets corresponding to the fractional B-splines in Fig. 1 is given in Fig. 3. They clearly interpolate the conventional B-spline wavelets (thicker lines), which are the only ones to be truly compactly supported. As the degree α increases, the functions converge to modulated Gaussian which are known to optimally time-frequency localized in sense of Eisenberg uncertainty principle. This limit behavior can be inferred from the general convergence theorem in [3]. In principle, the asymptotic form for a given α can be obtained by expressing β^α as the *n*-fold convolution of some function φ_0.

Everything that has just be said is directly applicable to the symmetric case as well. The corresponding fractional B-spline wavelets are all symmetric, which may present some practical advantages. They coincide with the traditional ones when the degree is even (but not when it is odd !).

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[1] : the wavelet is orthogonal to the scaling function (or its dilated versions), but not necessarily orthogonal to its own shifts.

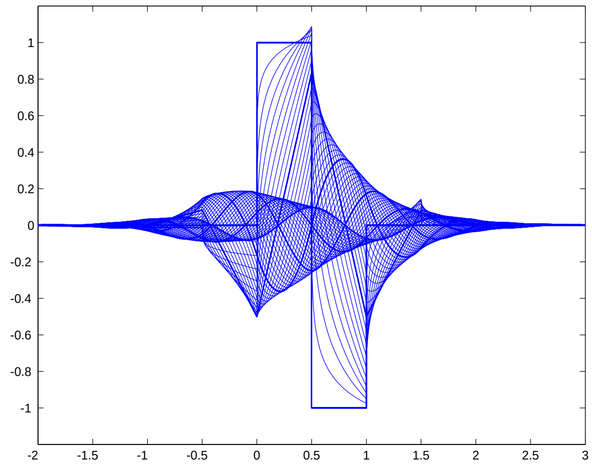

Fig. 3: The fractional B-spline wavelets with α≥0. These functions interpolate the conventional B-splines wavelets, which are represented using a thicker line.

4.3 Wavelet properties

The fractional splines wavelets of degree α, irrespective of their type, have $\lceil\alpha\rceil+1$ vanishing moments:

$$\int_{-\infty}^{+\infty} x^n \psi^\alpha(x)dx = 0, \quad n = 0,\ldots,\lceil\alpha\rceil \tag{37}$$

This comes as a direct consequence of the polynomial reproduction properties of the fractional B-splines and the fact that the scaling function and wavelets are orthogonal.

Another remarkable property is that the fractional spline wavelets behave like fractional derivative operators. Specifically, we can show that

$$\hat{\psi}_+^\alpha(\omega) = C \cdot (j\omega)^{\alpha+1}, \text{ as } \omega \to 0 \tag{38}$$

and

$$\hat{\psi}_*^\alpha(\omega) = C \cdot |\omega|^{\alpha+1}, \quad \text{as } \omega \to 0 \tag{39}$$

where $\psi_*^\alpha(x)$ is symmetric. This implies that for a slowly varying signal $f(x)$ ($\hat{f}$ essentially lowpass and concentrated around the origin), one has $\langle f, \psi^\alpha(x-y)\rangle \propto D^{\alpha+1} f(y)$, where D^α is the fractional derivative of order α. Thus, one potential application of these wavelets is the analysis or synthesis of fractional Brownian motion processes. In particular, such wavelets may be used to whiten fractal processes whose spectral power density decays like $1/\omega^{2H+1}$.

5. CONCLUSION

We extended the B-splines to fractional order. What is remarkable is that these new functions inherit all the nice properties of the polynomial B-splines with two exceptions: positivity and compact support. This allowed us to construct fractional wavelet bases of L_2. In particular, we described enlarged families of orthogonal and semi-orthogonal spline wavelets with a continuous order indexing rather than a discrete one.

These new fractional spline wavelets have explicit formulas in both the time and frequency domain. Their most notable feature is their order of approximation $\alpha+1$ which is no longer an integer. In addition, they behave like fractional derivative operators. Their only shortcoming is that they are not compactly supported. We found that the underlying wavelet filters decay reasonably fast for $\alpha > 0$ so that the numerical implementation of these transforms is not a problem.

A software demo is available at: `http://bigwww.epfl.ch/demo/fractsplines.`

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