A Complete Family of Scaling Functions: The (α,τ)-Fractional Splines

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Summary

We describe a new family of scaling functions, the (α,τ)-*fractional splines*, which generate valid multiresolution analyses. These functions are characterized by two real parameters: α, which controls the **width** of the scaling functions; and τ, which specifies their **position** with respect to the grid (shift parameter). This new family is complete in the sense that it is closed under convolutions and correlations.

We give the explicit time and Fourier domain expressions of these fractional splines.

We prove that the family is closed under generalized fractional differentiations, and, in particular, under the Hilbert transformation. We also show that the associated wavelets are able to whiten $1/f^\lambda$-type noise, by an adequate tuning of the spline parameters.

A fast (and exact) FFT-based implementation of the fractional spline wavelet transform is already available. We show that fractional integration operators can be expressed as the composition of an analysis and a synthesis iterated filterbank.

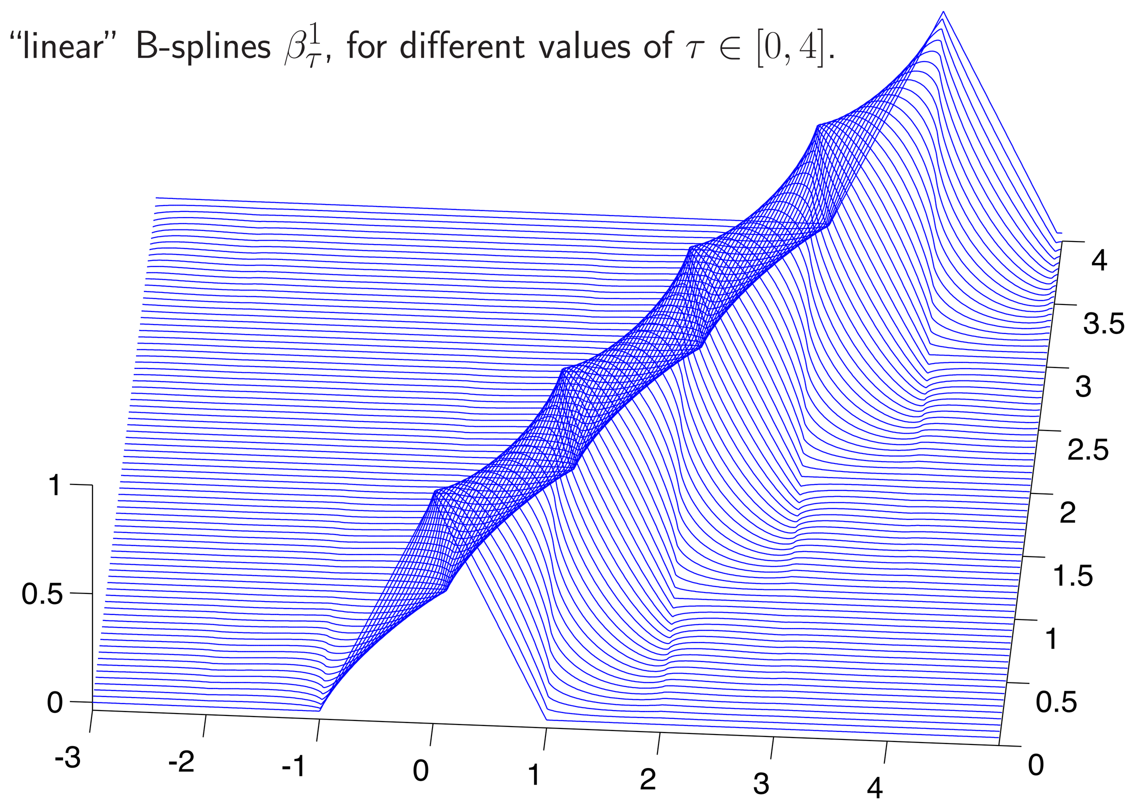
Expression of the Fractional Splines

Let $\alpha > -1$ and τ be some real parameters.

- **Fourier**:

$$\hat\beta^\alpha_\tau(\omega) = \left(\frac{e^{j\omega}-1}{j\omega}\right)^{\frac{\alpha+1}{2}-\tau}\left(\frac{1-e^{-j\omega}}{j\omega}\right)^{\frac{\alpha+1}{2}+\tau}$$

The "linear" B-splines β^1_τ, for different values of $\tau \in [0,4]$.



- **Time-domain**:

$$\beta^\alpha_\tau(t) = \sum_k (-1)^k \left|\begin{matrix}\alpha+1\\ k-\tau\end{matrix}\right| \rho^\alpha_\tau(t-k)$$

where (if α is not integer)

$$\rho^\alpha_\tau(t) = \text{Const} \times |t|^\alpha + \text{Const} \times |t|^\alpha \operatorname{sign}(t)$$

$$\left|\begin{matrix}p\\ q\end{matrix}\right| = \frac{\Gamma(p+1)}{\Gamma\left(\frac{p}{2}+q+1\right)\Gamma\left(\frac{p}{2}-q+1\right)}$$

If α is integer then, either $|t|^\alpha$ degenerates into $|t|^\alpha \log|t|$ (α even); or $|t|^\alpha \operatorname{sign}(t)$ degenerates into $|t|^\alpha \log|t| \operatorname{sign}(t)$ (α odd).

- **Asymptotics**: For large α, a fractional B-spline is a Gaussian

$$\beta^\alpha_\tau(t) \underset{\alpha\to\infty}{\approx} \sqrt{\frac{6}{\pi(\alpha+1)}}\, e^{-\frac{6}{\alpha+1}(t-\tau)^2}$$

of width $\sqrt{\frac{\alpha+1}{12}}$ centered at τ.

Properties

- **Two-scale difference equation**:

$$\beta^\alpha_\tau(t) = \sum_k \underbrace{2^{-\alpha}\left|\begin{matrix}\alpha+1\\ k-\tau\end{matrix}\right|}_{h^\alpha_\tau[n]} \beta^\alpha_\tau(2t-k)$$

which allows to build *wavelet* bases. Note: the scaling filter contains a non integer number of "regularity" factors

$$H^\alpha_\tau(z) = 2^{-\alpha}(1+z)^{\frac{\alpha+1}{2}-\tau}(1+z^{-1})^{\frac{\alpha+1}{2}+\tau}.$$

- **Fractional differentiation**: We first define the fractional derivative of a function $f(t)$ by

$$\partial^\alpha_\tau f(t) = \int_{-\infty}^{\infty} (-j\omega)^{\frac{\alpha}{2}-\tau}(j\omega)^{\frac{\alpha}{2}+\tau}\hat f(\omega)\frac{\mathrm{d}\omega}{2\pi}$$

Then, the fractional derivative of order (α',τ') of the (α,τ)-fractional B-spline is a *digitally-filtered* version of the $(\alpha-\alpha',\tau-\tau')$-fractional B-spline

$$\partial^{\alpha'}_{\tau'}\beta^\alpha_\tau(t) = \sum_k (-1)^k \left|\begin{matrix}\alpha'\\ k-\tau'\end{matrix}\right| \beta^{\alpha-\alpha'}_{\tau-\tau'}(t-k).$$

- **Hilbert transform**: remarking that the Hilbert transform operator is a fractional derivative of order 0

$$\mathscr{H} f = -\partial^0_{1/2} f$$

we get that the Hilbert transform of the (α,τ)-fractional B-spline is in the resolution space of the $(\alpha,\tau-1/2)$-fractional B-spline

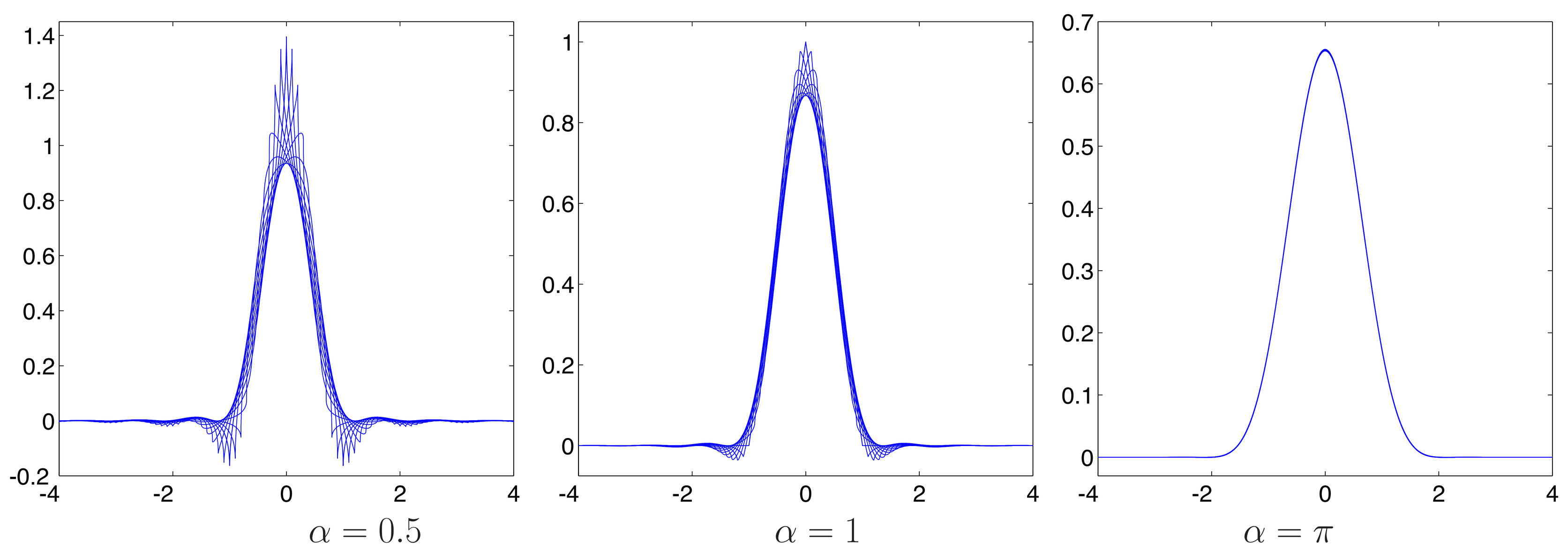
$$\mathscr{H}\beta^\alpha_\tau(t) = \sum_k \frac{1}{\pi(k-\frac12)}\beta^\alpha_{\tau-1/2}(t-k)$$

Interpretation of the parameters

- α is a polynomial degree. It is also related to the *effective support* of the B-spline ($\propto \sqrt{\alpha+1}$). α is its (Hölder) regularity order and $\alpha+1$ is its approximation order;
- τ is the center of the B-spline for higher degrees—hence behaves as a "shift"

$$\beta^\alpha_\tau(t) \approx \beta^\alpha_0(t-\tau) \quad \text{when } \alpha \geq 3;$$

τ is a dissymmetry coefficient for smaller degrees. Note: it is essentially a phase term in Fourier since $|\hat\beta^\alpha_\tau(\omega)| = \hat\beta^\alpha_0(\omega)$—independent of τ.



Plots of $\beta^\alpha_\tau(t+\tau)$, for three different values of α; on each plot, τ varies in $[0,1]$.

Wavelets

- **Fractional B-spline wavelets**: we first define the wavelet filter $G^\alpha_\tau(z)$ by

$$G^\alpha_\tau(z) = -z^{-1}H^\alpha_\tau(-z^{-1})A^\alpha(-z) = \sum_k g^\alpha_\tau[k]z^{-k}$$

where $A^\alpha(-z)$ is the autocorrelation filter of the (α,τ)-fractional B-spline

$$A^\alpha(z) = \sum_k z^{-k}\int \beta^\alpha_\tau(u+k)\beta^\alpha_\tau(u)\,\mathrm{d}u = \sum_k z^{-k}\beta^{2\alpha+1}_0(k).$$

Then, the fractional B-spline wavelet is obtained through the scaling relation

$$\psi^\alpha_\tau(t) = \sum_k g^\alpha_\tau[k]\,\beta^\alpha_\tau(2t-k)$$

This function is biorthogonal to $\beta^\alpha_\tau(t-k)$—and is the one that has the smallest support, when $\beta^\alpha_\tau(t)$ has bounded support—i.e., when α is integer and $\tau = \frac{\alpha+1}{2}$.

- **Discrete wavelet transform**: exact (for periodic boundary conditions), fast implementation via the FFT (cf. Blu/Unser ICASSP'00). Java demo available online at `http://bigwww.epfl.ch/demo/jfractsplinewavelet/index.html`.
Besides, the wavelet transform coefficient

$$\left\langle f, \psi^\alpha_\tau(2^{-i}t-k)\right\rangle \approx -2^{i(\alpha+3/2)}\frac{A^\alpha(\pi)}{4^{\alpha+1}}\partial^{\alpha+1}_\tau f(k2^i)$$

behaves like a generalized fractional derivative $\partial^{\alpha+1}_\tau$ of order $\alpha+1$, evaluated at the point $k2^i \rightsquigarrow$ whitening of $1/|\omega|^{\frac{\alpha+1}{2}}$-type noise.

Application: solving $\partial^{\alpha'}_{\tau'} f(t) = g(t)$

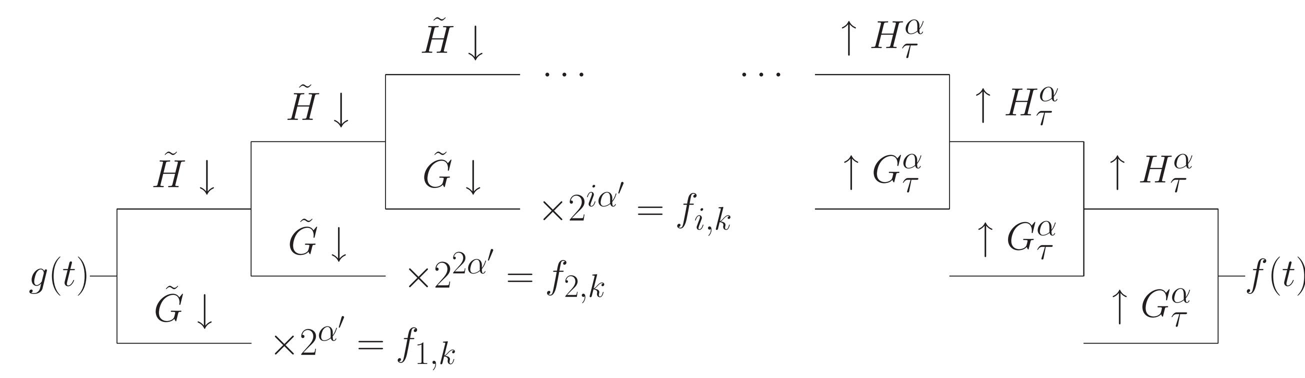
Example: $g(t) =$ white noise $\rightsquigarrow f(t) =$ **fractional Brownian motion**.
Let $f_{i,k}$ be the unknown coefficients of the (α,τ)- fractional spline wavelet decomposition of $f(t)$; then

$$g(t) = \sum_{i,k} f_{i,k}2^{-i\alpha'}\partial^{\alpha'}_{\tau'}\psi^\alpha_\tau(2^{-i}t-k)$$

which points out that the coefficients $f_{i,k}$ are given by the *wavelet decomposition of $g(t)$ using the wavelet $\partial^{\alpha'}_{\tau'}\psi^\alpha_\tau$*. For this decomposition, the associated scaling function is $\beta^{\alpha-\alpha'}_{\tau-\tau'}(t)$ of scaling filter $H(z) = H^{\alpha-\alpha'}_{\tau-\tau'}(z)$; the wavelet filter corresponding to $\partial^{\alpha'}_{\tau'}\psi^\alpha_\tau$ is thus $G(z) = 2^{2\alpha'-1}H^{\alpha'-1}_{\tau'}(-z)G^\alpha_\tau(z)$. This implies that the coefficients of the decomposition are obtained via the analysis filters:

$$\tilde H(z) = H^{\alpha+\alpha'}_{\tau'-\tau}(z)\frac{A^\alpha(z)}{A^\alpha(z^2)} \quad \text{and} \quad \tilde G(z) = -\frac{2^{-2\alpha'+1}zH^{\alpha-\alpha'}_{\tau-\tau'}(-z)}{A^\alpha(z^2)}$$

$f(t)$ is finally obtained by multiplying these coefficients by $2^{i\alpha'}$ and performing an (α,τ)-fractional spline wavelet synthesis.



Resolution of the differential equation $\partial^{\alpha'}_{\tau'} f = g$ using a dyadic analysis-synthesis filterbank.