Rethinking Super-resolution: The Bandwidth Selection Problem

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Summary

Key Takeaways

- Super-resolution is the art of recovering spikes from their low-pass projections.
- Over the last decade specifically, several significant advancements linked with mathematical guarantees and recovery algorithms have been made.
- Most super-resolution algorithms rely on a two-step procedure: deconvolution followed by high-resolution frequency estimation.
- However, for this to work, exact bandwidth of low-pass filter must be known; an assumption that is central to the mathematical model of super-resolution.
- On the flip side, when it comes to practice, smoothness rather than bandlimitedness is a much more applicable property.
- Since smooth pulses decay quickly, one may still capitalize on the existing super-resolution algorithms provided that the essential bandwidth is known.
- This problem has not been discussed in literature and is the theme of our work.
- We propose a bandwidth selection criterion which works by minimizing a proxy of estimation error that is dependent of bandwidth.

Setup for Super-resolution of Sparse Signals

Given N time-domain, sampled measurements, $y(nT)$ of the continuous signal

$$y(t) = \sum_{k=0}^{K-1} c_k \phi(t - t_k), \tag{1}$$

the super-resolution problem seeks to recover the $2K$ unknowns $\{c_k, t_k\}_{k=0}^{K-1}$ assuming that: (A1) K and ϕ are known; and (A2) ϕ is bandlimited (its Fourier transform is compactly supported). The notion of sparsity naturally finds its way in the super-resolution problem because $y(t) = (\phi * s)(t)$ where s is a continuous-time, K-sparse signal

$$s(t) = \sum_{k=0}^{K-1} c_k \delta(t - t_k), \quad t_k \in [0, \tau). \tag{2}$$

Recovery Strategy

Typical recovery procedure in the super-resolution problem exploits the structure of sparse signal. This is done in two steps:

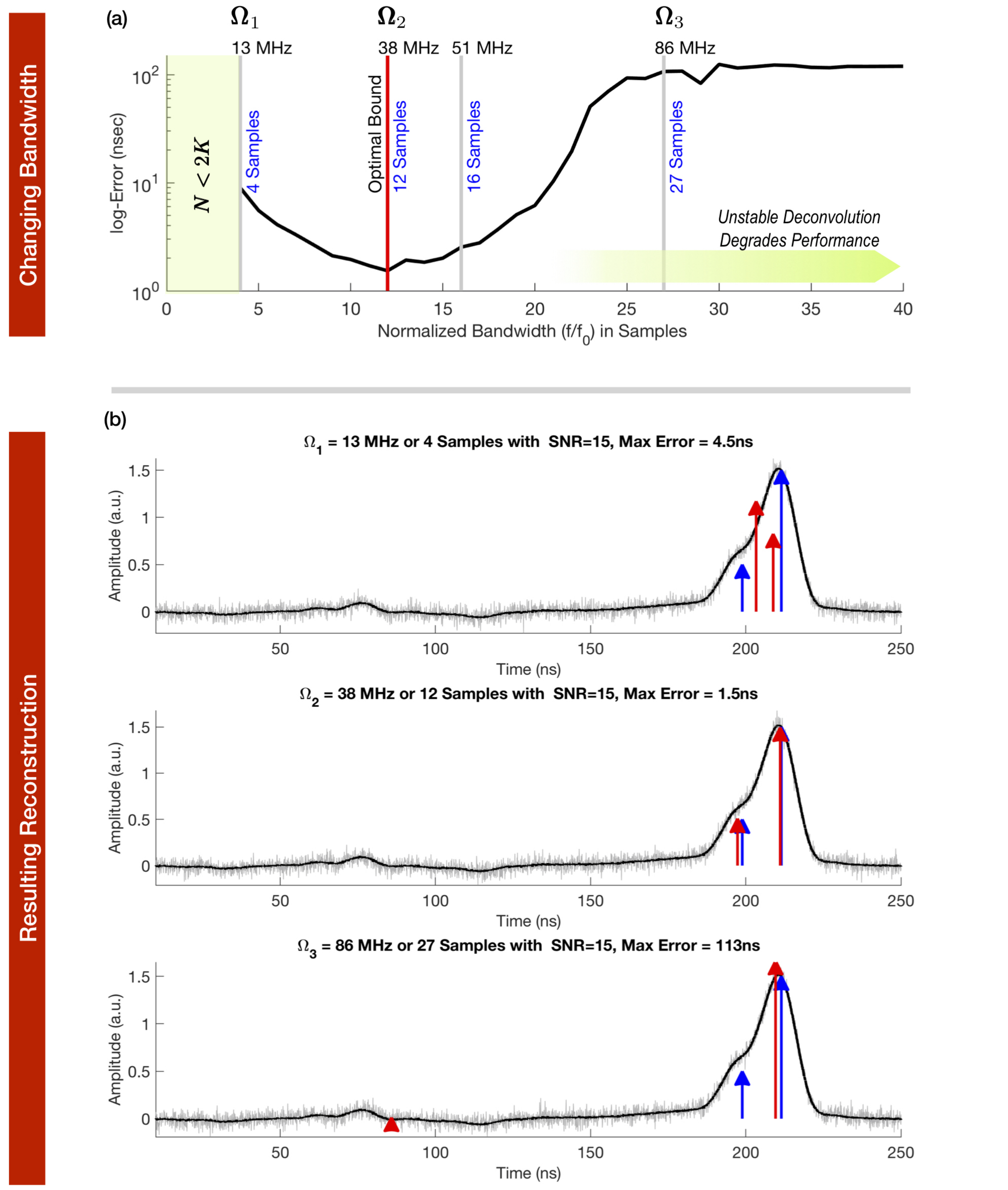
1. **Deconvolution**.
Here $\widehat{s}(n\omega_0)$ is estimated by using,

$$\widehat{s}(n\omega_0) = \frac{\widehat{y}(n\omega_0)}{\widehat{\phi}(n\omega_0)} = \sum_{k=0}^{K-1} c_k e^{-\jmath n\omega_0 t_k}, \qquad n\omega_0 \in [-\Omega, \Omega]$$

where Ω is the bandwidth of ϕ.

2. **Parameter Estimation**.
Once $\widehat{s}(n\omega_0)$ is computed, its parametric/sinusoidal form is then used for estimating unknowns $\{c_k, t_k\}_{k=0}^{K-1}$ using high resolution spectral estimation methods, fitting approaches or recently developed convex-optimization based approaches.

Super-resolution is Sensitive to Bandwidth



Bandwidth Affects Reconstruction

Varying Ω arbitrarily, leads to the following scenarios.

- When Ω is such that $N < 2K$, the parameter estimation by fitting will fail as the system is under-determined.
- Gradually increasing Ω such that $2K\omega_0 \leq \Omega < \Omega_0$ leads to *over-sampling* and hence to performance enhancement of the spectral estimation methods.
- Understandably, when Ω approaches the heuristically chosen Ω_0, the deconvolution step becomes ill-posed.

Towards a Bandwidth Selection Principle

Typically, in practice, ϕ is smooth and the selection criterion for bandwidth parameter Ω is *unclear*. Consider the case of noisy measurements $m(t) = y(t) + e(t)$ where $e(t)$ is bounded noise. Dividing $\widehat{m}(\omega)$ by $\widehat{\phi}$ (i.e. deconvolving), we obtain

$$\frac{\widehat{m}(\omega)}{\widehat{\phi}(\omega)} = \sum_{k=0}^{K-1} c_k e^{-\jmath \omega t_k} + \widehat{e}_\phi(\omega), \quad |\omega| \leqslant \Omega \tag{3}$$

$$|\widehat{e}_\phi(\omega)| = \left|\frac{\widehat{e}(\omega)}{\widehat{\phi}(\omega)}\right| \leqslant \eta \cdot \underbrace{\left(\min_{|\omega|\leqslant\Omega} \left|\widehat{\phi}(\omega)\right|\right)^{-1}}_{:=\varepsilon_\Omega}. \tag{4}$$

The bandwidth selection criterion is given by, $\Omega_{\text{opt}} = \arg\min_\Omega \mathsf{G}(\Omega, \mathcal{D})\varepsilon_\Omega$.

In the above, $\mathsf{G}(\Omega, \mathcal{D})$ upper-bounds a quantity *linearized condition number* $\kappa^{(\ell)}$,

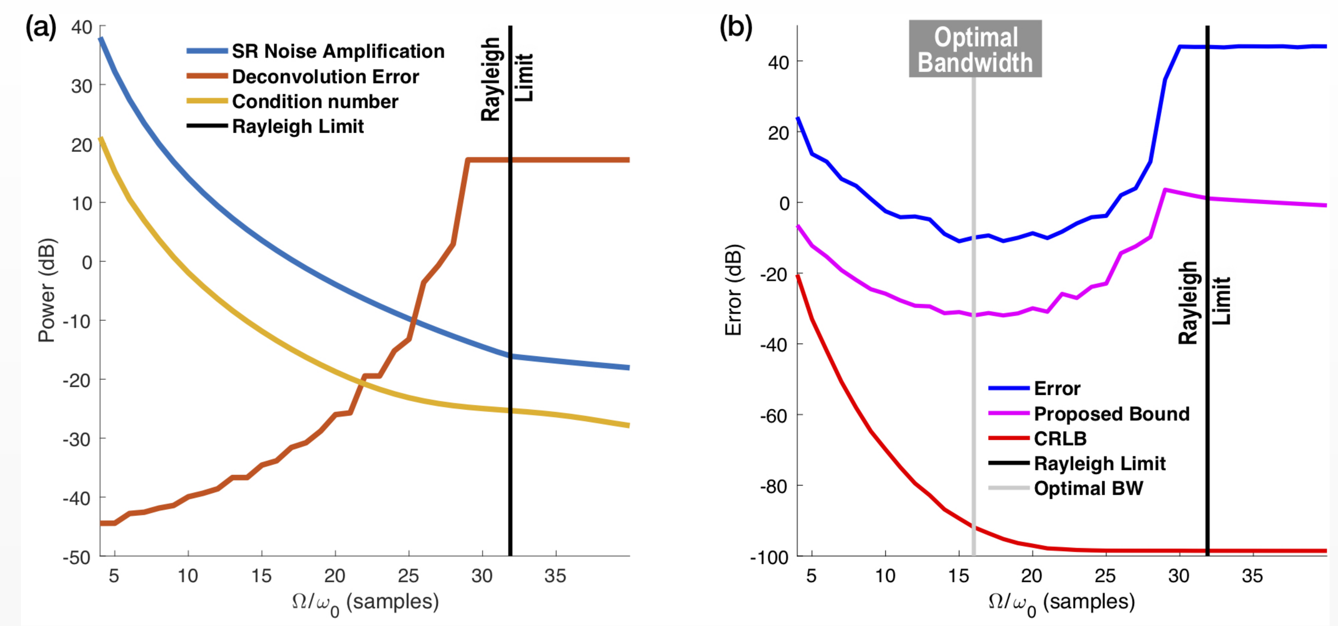
$$\sup_{\underline{\theta}\in\mathcal{D}, k\in[0,K-1]} \kappa^{(2k+1)}(\underline{\theta}, \Omega) \leqslant \mathsf{G}(\Omega, \mathcal{D}), \qquad \underline{\theta} := \{c_k, t_k\}_{k=0}^{K-1} \in \mathbb{R}^{2K}.$$

More precisely, $\kappa^{(m)}$ is the ℓ_1 norm of the m-th row of the matrix $\mathbf{J}^\dagger$, where $\mathbf{J}$ is the Jacobian matrix representing $\widehat{s}(n\omega_0)$, and $(\cdot)^\dagger$ is the Moore-Penrose pseudo-inverse.

Theorem: Suppose that $\forall \underline{\theta} \in \mathcal{D} \subset \mathbb{R}^{2K}$, the amplitudes are bounded: $0 < A_1 \leq |c_k| \leq A_2$, and the minimal distance $\mathsf{M}_\delta = \min_{k\neq\ell} |t_k - t_\ell| \geqslant \Delta > 0$ is also bounded. There exist constants $\{C_k\}_{k=1}^3$, depending on A_1, A_2, K, such that the following bounds hold.

- Well-separated Regime: If $\Delta > C_1/\Omega$, then $\kappa^{(\ell)} \leqslant C_2/\Omega, \ell = 1, 3, \ldots, 2K-1$.
- Single Cluster Regime: If $\mathsf{M}_\delta < 2\pi K/\Omega$, then $\kappa^{(\ell)} \leqslant (C_3/\Omega)(\Omega\Delta)^{2K-2}$.

Optimal Bandwidth Computation



- L. Li and T. P. Speed, “Parametric deconvolution of positive spike trains,” *Annals of Statistics*, pp. 1279–1301, 2000.
- M. Vetterli, P. Marziliano, and T. Blu, “Sampling signals with finite rate of innovation,” *IEEE Trans. Signal Process.*, 2002.
- E. Candès and C. Fernandez-Granda, “Towards a mathematical theory of super-resolution,” *Communications on Pure and Applied Mathematics*, vol. 67, no. 6, pp. 906–956, 2014.
- D. Batenkov, “Stability and super-resolution of generalized spike recovery,” *Applied and Computational Harmonic Analysis*, vol. 45, no. 2, pp. 299–323, Sep. 2018.
- D. Batenkov, G. Goldman and Y. Yomdin, “ Super-resolution of near-colliding point sources,” *arXiv preprint* (arXiv:1904.09186), 2019.