Why restrict ourselves to compactly supported basis functions?

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Abstract

Compact support is undoubtedly one of the wavelet properties that is given the greatest weight both in theory and applications. It is usually believed to be essential for two main reasons : (1) to have fast numerical algorithms, and (2) to have good time or space localization properties. Here, we argue that this constraint is unnecessarily restrictive and that fast algorithms and good localization can also be achieved with non-compactly supported basis functions. By dropping the compact support requirement, one gains in flexibility. This opens up new perspectives such as fractional wavelets whose key parameters (order, regularity, etc...) are tunable in a continuous fashion. To make our point, we draw an analogy with the closely related task of image interpolation. This is an area where it was believed until very recently that interpolators should be designed to be compactly supported for best results. Today, there is compelling evidence that non-compactly supported interpolators (such as splines, and others) provide the best cost/performance tradeoff.

I. Introduction

A good part of the effort in wavelet theory has been directed towards the design of compactly supported basis functions. The key argument is that such wavelet transforms can be implemented exactly using FIR filters and that they give rise to fast algorithms.

In this presentation, we will argue that the constraint of compact support is unnecessarily restrictive and that one can achieve more by considering extended classes of basis functions. We will also dispel the commonly-held belief that using non-compact basis functions is computationally not effective. In particular, we will demonstrate an exact FFT-based implementation of the Mallat algorithm which can be quite competitive with the standard algorithm, especially in higher dimensions.

Interestingly, non-compactly supported basis functions bring us back to the very beginning of the field with the Meyer[8] and Battle-Lemarié[1, 6] wavelets. These are worth mentioning because they display two properties—orthogonality and symmetry—which cannot be achieved simultaneously with compactly supported wavelets[4]. Working with symmetric or anti-symmetric basis functions makes perfect sense for image processing because they do not modify the location of edges which are the main clues for object recognition.

While symmetry + orthogonality can be part of the argument, there are also other stronger reasons for considering non-compactly supported basis functions. First, we believe that working with non-compactly supported basis function can provide better cost/performance tradeoffs. While we don't yet have a wavelet example to demonstrate this point, we can look at the closely related problem of interpolation which has very much the same mathematical flavor. This is what we will do in Section II, where we will present recent research results that unambiguously demonstrate the superiority of interpolators with infinite impulse responses—more precisely, mixed schemes for which the non-compact part is implemented via IIR digital filtering. The second argument is increased flexibility and tunability. Here the idea is to have access to wavelet transforms with an adjustable knob which controls the shape and key properties of the basis functions. In Section III, we will consider the interesting case of the fractional wavelet transforms and also demonstrate how they can be implemented very efficiently.

II. Interpolation: IIR is usually better

Let us consider the interpolation problem where the task is to fit the uniform samples $f(k)$ of a signal with a wavelet-like model which uses the integer shifts of some basis function $\phi(x)$. Here, the traditional approach in image processing has been to design compactly supported basis functions that are interpolating so that the signal samples can be used as expansion coefficients directly. The primary motivation there was to simplify computation because the model fitting step is essentially free. The question, however, is to see if this really pays off globally when the interpolation model is used to re-sample the signal at non-integer locations. There is now compelling evidence that this is not the optimal approach. Recent studies have shown that, for a given computational cost, it is more efficient to work with non-interpolating basis functions such as the B-splines[7, 9] and the Omoms[2] (optimal maximum order and minimum support). These can be fitted to the signal samples at very low cost using recursive digital filtering as described elsewhere[10]. This results in an equivalent interpolator whose impulse response is no longer compactly supported even though the re-sampling is implemented using compactly supported basis functions. What achieves the trick is the factorization of the interpolation process in two parts: the specification of a model that uses the shortest possible basis functions with the best approximation properties (e.g., B-splines[10] or some close relatives[2]), and the recursive digital filtering which implements the infinite interpolator at a finite cost. For image processing applications, the cost of prefiltering is essentially negligible so that it really makes sense to optimize the basis functions for best performance. The key idea here that can be transposed to wavelets as well is that it may be advantageous to work with very short basis functions on the synthesis side (where you really need them), but not necessarily on the analysis side. In other words, it is possible to compute inner products with non-compactly supported basis functions at low cost via recursive digital filtering[5, 11].

III. Fractional wavelet bases

Fractional wavelets[13] constitute extended families of wavelet bases indexed by a continuously-varying order parameter γ. They can be viewed as some kind of interpolation of the traditional families which are specified for integer orders only (number of vanishing moments). Having access to intermediate transforms with γ non-integer brings flexibility: it means that we can control the key properties of the transform such as order of approximation, regularity, and localization of the basis functions, which may be useful for some applications. Fractional wavelets also behave like fractional derivative operators, which makes them well suited for dealing with fractal and fractional Brownian-motion-like processes.

While our first example involved spline basis functions[12], we can define fractional wavelets in more general terms. Specifically, we will say that $\varphi(x)$ is a valid *fractional* scaling function of order γ if and only if:

(*i*) it generates a valid Riesz basis; i.e., there exist $A > 0$ and $B < +\infty$ such that

$$c \in l_2, \quad A\,\|c\|_{l_2}^2 \le \left\|\sum_{k \in Z} c(k)\varphi(x-k)\right\|^2 \le B\,\|c\|_{l_2}^2 \tag{1}$$

(*ii*) its refinement filter can be factorized as

$$H_\gamma(z) = \left(\frac{1+z^{-1}}{2}\right)^\gamma Q(z) \tag{2}$$

where $Q(z)$ stable; i.e., $|Q(e^{j\omega})| < +\infty$. This is similar to the conventional definition of a scaling function, with one crucial difference: $\gamma \in R^+$ is not necessarily integer anymore. Otherwise, there is not much change for the practitioner: the transform is still implemented using a two channel perfect reconstruction filterbank; the wavelet filters can also be specified to yield various types of decompositions; i.e., orthogonal, semi-orthogonal, or biorthogonal. In fact, it is possible—at least in principle—to extend all conventional wavelet families to fractional orders.

In the case where γ is fractional—i.e., non-integer—we can use the generalized binomial theorem to expand the order-determining factor of the filter as:

$$\left(1+z^{-1}\right)^\gamma = \sum_{k=0}^{+\infty} \binom{\gamma}{k} z^{-k} \tag{3}$$

This sum is infinite unless γ is integer; it involves the generalized binomial coefficients

$$\binom{u}{v} = \frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)} \tag{4}$$

where $\Gamma(u+1) = u!$ is Euler's Gamma function which generalizes the factorial. Thus, it follows that $H_\gamma(z)$ cannot be compactly supported when γ is non-integer, which means that the corresponding scaling function and wavelet will be infinitely supported as well.

At first sight, the fractional wavelets may look like a computational nightmare because they are not compactly supported. Truncation of the filters is problematic because these are typically specified in the frequency domain; also, getting a good mathematical handle on their time-decay is much more difficult than in the traditional case. Fortunately, one can turn the situation around by implementing the transforms in the frequency domain using FFTs. This turns out to be quite efficient computationally but also very simple to code[3]. It automatically takes care of boundary conditions and also makes the specification of the wavelet filters quite straightforward. Another advantage is that the code is generic; it works for any value of γ.

Fractional wavelet software, art and demonstrations are available at: http://bigwww.epfl.ch/demo/fractsplines/.

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