ON THE APPROXIMATION POWER OF SPLINES: ORTHOGONAL VERSUS HEXAGONAL LATTICES

Dimitri Van De Ville, Thierry Blu, Michael Unser

Biomedical Imaging Group STI / BIO-E
Swiss Federal Institute of Technology Lausanne (EPFL)
CH-1015 Lausanne EPFL, Switzerland

Extended abstract

Recently, we have proposed a novel family of bivariate, non-separable splines [1]. These splines, called "hex-splines" have been designed to deal with hexagonally sampled data. Incorporating the shape of the Voronoi cell of a hexagonal lattice, they preserve the twelve-fold symmetry of the hexagon tiling cell. Similar to B-splines, we can use them to provide a link between the discrete and the continuous domain, which is required for many fundamental operations such as interpolation and resampling [2]. The question we answer in this paper is "How well do the hex-splines approximate a given function in the continuous domain?" and more specifically "How do they compare to separable B-splines deployed on a lattice with the same sampling density?"

A general signal space, spanned by shifted versions of a function $\varphi(\mathbf{x})$ (such as a spline) on a lattice described by a matrix $\mathbf{R} = [\mathbf{r}_1 \mathbf{r}_2]$, contains all signals

$$s(\mathbf{x}) = \sum_{\mathbf{k}} c(\mathbf{k})\varphi(\mathbf{x} - \mathbf{R}\mathbf{k});\ c(\mathbf{k}) \in l_2(\mathbb{Z}^2). \qquad (1)$$

In general, the coefficients $c(\mathbf{k})$ are determined as

$$c(\mathbf{k}) = \int g(\mathbf{x})\tilde{\varphi}(\mathbf{x} - \mathbf{R}\mathbf{k})\mathrm{d}\mathbf{x}, \qquad (2)$$

where g is the original function and $\tilde{\varphi}$ is the prefilter. The optimal choice, i.e., corresponding to an orthogonal projection into the function space, is the dual filter $\hat{\tilde{\varphi}}_d = \hat{\varphi}/\hat{a}_\varphi$. Here $\hat{a}_\varphi$ is the Fourier transform of the sampled autocorrelation function of φ. Another common choice is the interpolation prefilter, which selects $c(\mathbf{k})$ such that $s(\mathbf{R}\mathbf{k}) = g(\mathbf{R}\mathbf{k})$.

Separable B-splines are a perfect fit to be used as basis functions on conventional rectangular lattices. For hexagonal lattices, one can use a "slanted" version of the B-splines. Their support correspond to a rhomboid. Recently, we proposed the use of hex-splines, which are inspired on the Voronoi cell indicator function and exhibit a hexagonal support. Higher order hex-splines are constructed by successive two-dimensional convolutions. First, we want to compare hex-splines versus slanted B-splines on the same hexagonal lattice. Second, we also compare hex-splines on a hexagonal lattice against separable B-splines on a square lattice with the same sampling density. These comparisons are done from an approximation theory point of view.

Approximation theory provides us with a convenient way to quantify the approximation error by integration with an error kernel $E(\boldsymbol{\omega})$ in the Fourier domain: [3]

$$||s(\mathbf{x}) - g(\mathbf{x})||^2 = \frac{1}{4\pi^2}\int |\hat{g}(\boldsymbol{\omega})|^2 E(\boldsymbol{\omega})\mathrm{d}\boldsymbol{\omega}. \qquad (3)$$

This error kernel is composed out of two parts:

$$\begin{aligned} E(\boldsymbol{\omega}) &= E_{\mathrm{min}} + E_{\mathrm{res}} \\ &= 1 - \frac{|\hat{\varphi}(\boldsymbol{\omega})|^2}{\hat{a}_\varphi(\boldsymbol{\omega})} + E_{\mathrm{res}}. \end{aligned}$$

Most important, in the case of using the optimal prefilter (orthogonal projection), this kernel reduces to E_{min}.

The asymptotic behavior tells us how well the approximation converges to the original g when the sampling lattice is made finer by a scaling factor h. In this case, the argument of the error kernel under the integral of Eq. (3) is scaled accordingly as $E(h\boldsymbol{\omega})$. So by analyzing $E(\boldsymbol{\omega})$ around $\mathbf{0}$ we obtain

$$\begin{aligned} ||s(\mathbf{x}) - g(\mathbf{x})||^2 &\propto E(h\boldsymbol{\omega}) \\ &\propto h^{2L} O(||\boldsymbol{\omega}||^{2L}), \end{aligned} \qquad (4)$$

when L is the order of approximation. The constants in front of $\omega_1^k \omega_2^{2L-k}$ allow us to compare the behavior of $E(\boldsymbol{\omega})$ between different signal spaces when the order of approximation is the same.

After a brief introduction on hex-splines and approximation theory, this paper concentrates on orthogonal projection, i.e., using the optimal prefilter $\tilde{\varphi}_d$. As mentioned before, this is the best possible way to approach a function in the spline space. First, we compute the asymptotic constants for the hex-splines; i.e., the accurate asymptotic behavior of

(4) when the sampling grid gets denser. Second, we compare the asymptotic constants against the ones we obtain for slanted B-splines (with the same order of approximation L as the hex-splines) on the same lattice and B-splines (again the same order) on a square lattice with the same sampling density. As result, we find that the asymptotic constants for the hex-splines are smaller than those for the B-splines in both cases. Therefore, hex-splines are asymptotically a better representation than splines on hexagonal lattices. Second, hex-splines on a hexagonal lattice allow to obtain a better approximation for a given function than B-splines on an orthogonal lattice with the same sampling density.

References

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