SURE-LET Image Deconvolution using Multiple Wiener Filters

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Summary

We propose a novel deconvolution algorithm based on the minimization of Stein's unbiased risk estimate (**SURE**). We linearly parametrize the deconvolution process by using **multiple Wiener filterings** as elementary functions, followed by **undecimated Haar-wavelet thresholding**. The key contributions of our approach are: 1) the **linear combination** of several Wiener filters with different (but fixed) regularization parameters, which avoids the manual adjustment of a single nonlinear parameter; 2) the use of linear parameterization, which makes the SURE minimization finally boil down to **solving a linear system of equations**, leading to a very fast and exact optimization of the whole deconvolution process.

Problem statement

Linear observation model: $\mathbf{y} = \mathbf{Hx} + \mathbf{n}$

where $\mathbf{H}$ is the convolution matrix, Gaussian noise $\mathbf{n} \sim \mathcal{N}(0, \sigma^2\mathbf{I})$.

Problem: How to estimate $\mathbf{x}$ *from the observations* $\mathbf{y}$*, knowing* $\mathbf{h}$*?*

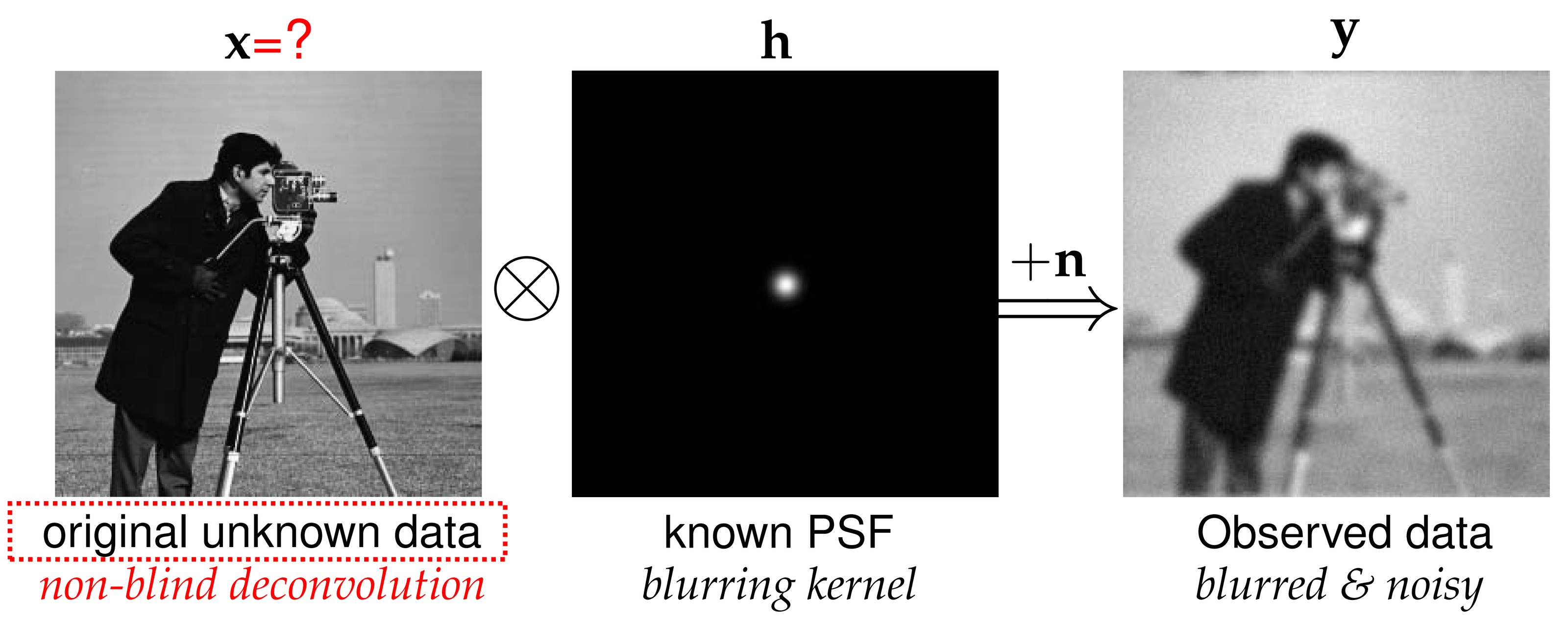


Figure 1: Deconvolution — estimation of original signal $\mathbf{x}$ from the distorted data $\mathbf{y}$.

SURE for deconvolution problems

Formulation — minimization of MSE

Denoting the processing of the measure data $\mathbf{y}$ by $\mathbf{F}$, our objective is to minimize the mean squared error (MSE):

$$\text{MSE} = \frac{1}{N}\mathscr{E}\{\|\underbrace{\mathbf{F}(\mathbf{y})}_{\text{the estimated data — the outcome of the processing } \mathbf{F}} - \mathbf{x}\|^2\}$$

SURE — unbiased estimate of MSE

Theorem Given the linear model above, the following random variable:

$$\epsilon = \frac{1}{N}\Big\{\|\mathbf{F}(\mathbf{y})\|^2 - 2\mathbf{y}^{\mathrm{T}}\mathbf{H}^{-\mathrm{T}}\mathbf{F}(\mathbf{y}) + 2\sigma^2 \mathrm{div}_{\mathbf{y}}\left(\mathbf{H}^{-\mathrm{T}}\mathbf{F}(\mathbf{y})\right)\Big\} + \underbrace{\frac{1}{N}\|\mathbf{x}\|^2}_{\text{neutral w.r.t. optimization}}$$

is an unbiased estimator of the MSE, i.e. $\mathscr{E}\{\epsilon\} = \frac{1}{N}\mathscr{E}\left\{\|\mathbf{F}(\mathbf{y}) - \mathbf{x}\|^2\right\}$, where the divergence operator is $\mathrm{div}_{\mathbf{y}}\mathbf{u} = \sum_{n=1}^{N}\frac{\partial u_n}{\partial y_n}$ for $\forall \mathbf{u} \in \mathbb{R}^N$.

The SURE-LET approach

Regularized SURE — an approximation of SURE

Considering the possible ill-posedness of the matrix $\mathbf{H}$, we approximate $\mathbf{H}^{-1}$ by a Tikhonov regularized inverse $\mathbf{H}_\beta^{-1}$:

$$\epsilon_\beta = \frac{1}{N}\Big\{\|\mathbf{F}(\mathbf{y})\|^2 - 2\mathbf{y}^{\mathrm{T}}\mathbf{H}_\beta^{-\mathrm{T}}\mathbf{F}(\mathbf{y}) + 2\sigma^2\mathrm{div}\left(\mathbf{H}_\beta^{-\mathrm{T}}\mathbf{F}(\mathbf{y})\right)\Big\} + \frac{1}{N}\|\mathbf{x}\|^2$$

where $\mathbf{H}_\beta^{-1} = \left(\mathbf{H}^{\mathrm{T}}\mathbf{H} + \beta\mathbf{S}^{\mathrm{T}}\mathbf{S}\right)^{-1}\mathbf{H}^{\mathrm{T}}$ for some β and matrix $\mathbf{S}$, to stabilize ϵ. In this work, we choose $\beta = 1 \times 10^{-5}\sigma^2$ and $\mathbf{S}$ as Laplacian operator.

Linear parametrization of the processing F — LET

The processing $\mathbf{F}(\mathbf{y})$ is represented by a **linear combination** of a small number ($K \ll N$) of known basic processings $\mathbf{F}_k(\mathbf{y}) \in \mathbb{R}^N$, weighted by unknown linear coefficients a_k for $k = 1, 2, ..., K$, i.e.

$$\mathbf{F}(\mathbf{y}) = \sum_{k=1}^{K} a_k \mathbf{F}_k(\mathbf{y})$$

The SURE-LET optimization

- Combining SURE and LET, the minimization of ϵ_β over the unknown linear weights a_k boils down to **solving a linear system of equations of order** K:

$$\sum_{k'=1}^{K}\underbrace{\left(\mathbf{F}_k^{\mathrm{T}}(\mathbf{y})\mathbf{F}_{k'}(\mathbf{y})\right)}_{\mathbf{M}_{k,k'}}a_{k'} - \underbrace{\left(\mathbf{y}^{\mathrm{T}}\mathbf{H}_\beta^{-\mathrm{T}}\mathbf{F}_k(\mathbf{y}) - \sigma^2\mathrm{div}\left(\mathbf{H}_\beta^{-\mathrm{T}}\mathbf{F}_k(\mathbf{y})\right)\right)}_{c_k} = 0$$

$$\Updownarrow$$

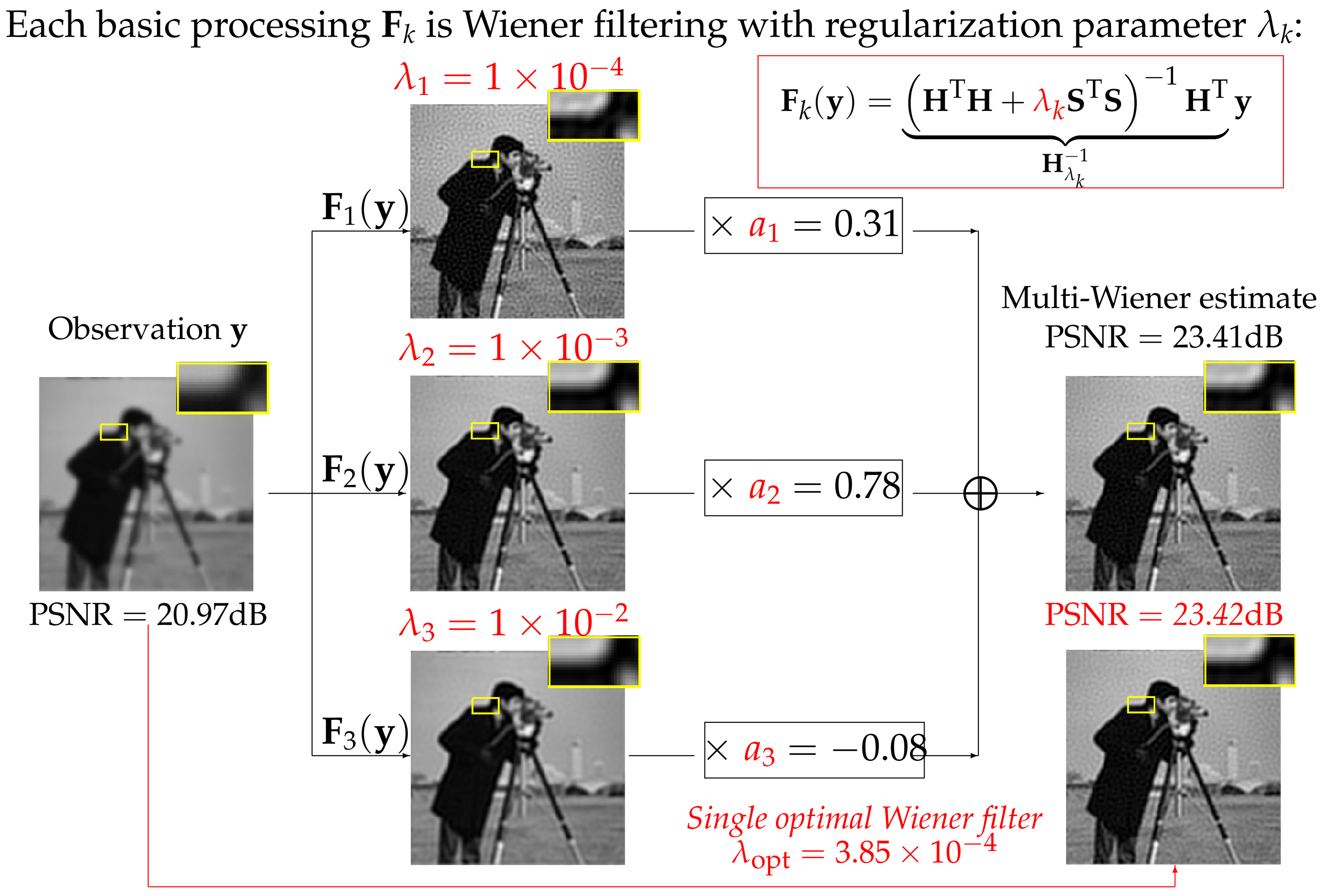
$$\mathbf{Ma} = \mathbf{c}$$

- **Advantage** of SURE-LET approach:
  1. dramatically reduce the deconvolution problem size from pixel number N to the number of basis functions K;
  2. simplify the deconvolution problem to solving a linear system of equations.

Construction of the functions $\mathbf{F}_k(\mathbf{y})$

Multi-Wiener deconvolutions

Each basic processing $\mathbf{F}_k$ is Wiener filtering with regularization parameter λ_k:



Multi-Wiener wavelet-thresholding deconvolution

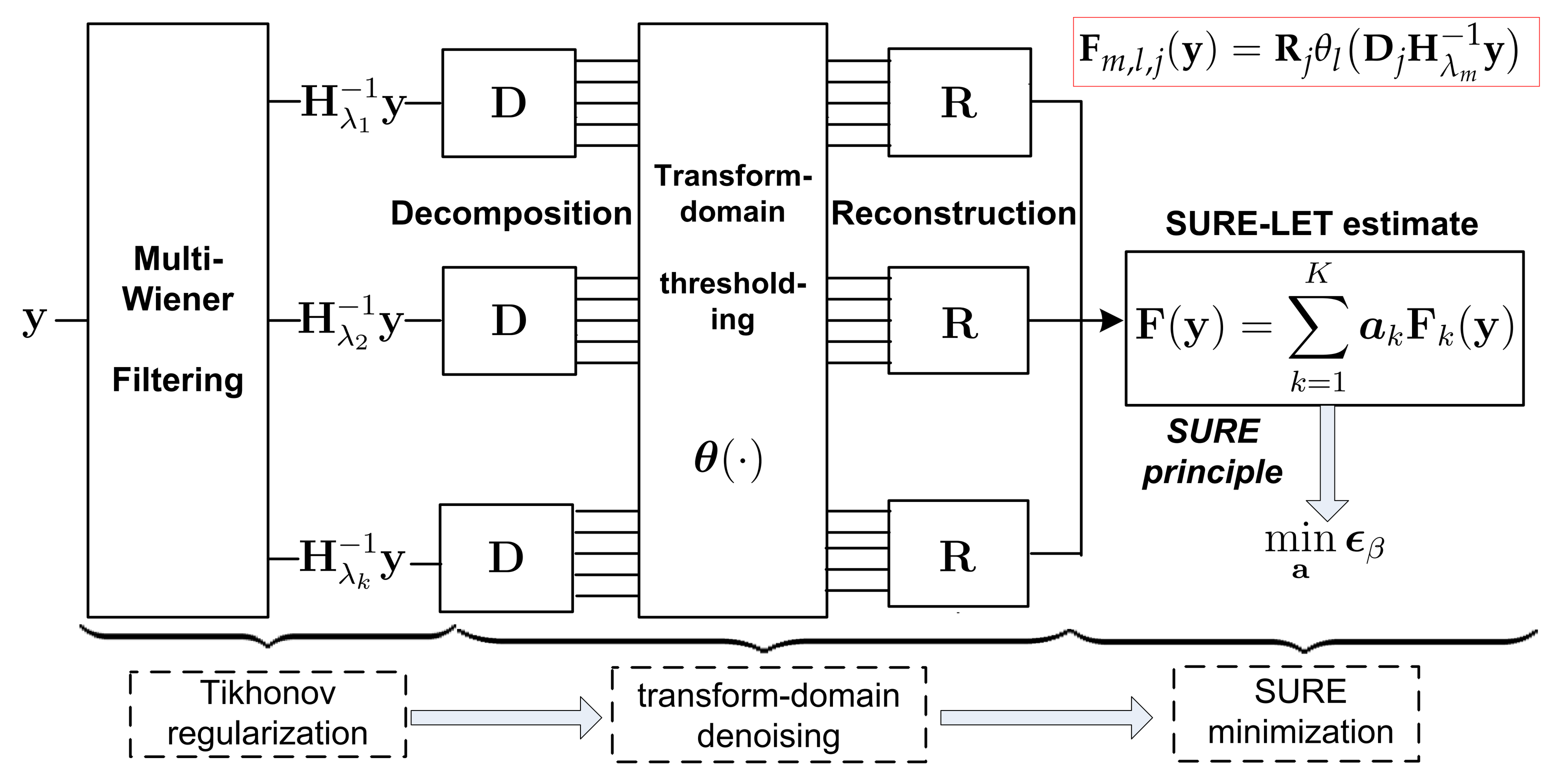


Figure 2: Typical structure of processing: multi-Wiener filtering followed by transform-domain thresholding, where the thresholding function is given as $\theta_l(w) = w\left\{1 - \exp\left(-\left(\frac{w}{T_l}\right)^4\right)\right\}$.

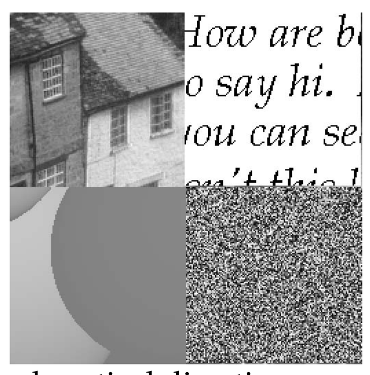
Experimental results

Parameter setting of the proposed SURE-LET algorithm

- λ_m: $\lambda_1 = 1 \times 10^{-4}\sigma^2$, $\lambda_2 = 1 \times 10^{-3}\sigma^2$, $\lambda_3 = 1 \times 10^{-2}\sigma^2$
- T_l: $T_1 = 4\sigma_{m,j}$, $T_2 = 9\sigma_{m,j}$
- $\mathbf{D}$ and $\mathbf{R}$: Haar wavelet
- $K = MJL + M$

Image *Mixture* deconvolution performance (PSNR in dB)

Blur	*Separable filter*			*9 × 9 uniform blur*		
σ	1	10	50	1	10	50
Input	18.38	17.94	12.76	14.58	14.40	11.35
BM3D	26.54	20.04	16.15	20.66	16.01	14.60
TVMM	27.17	20.64	15.25	20.70	15.64	13.66
C-SALSA	26.58	20.16	16.19	20.04	16.30	14.29
SURE-LET	**28.08**	**21.18**	**16.94**	**21.70**	**16.65**	**15.01**



⋆ *Separable filter*: with weights [1, 4, 6, 4, 1]/16 along both horizontal and vertical directions.

Visual example

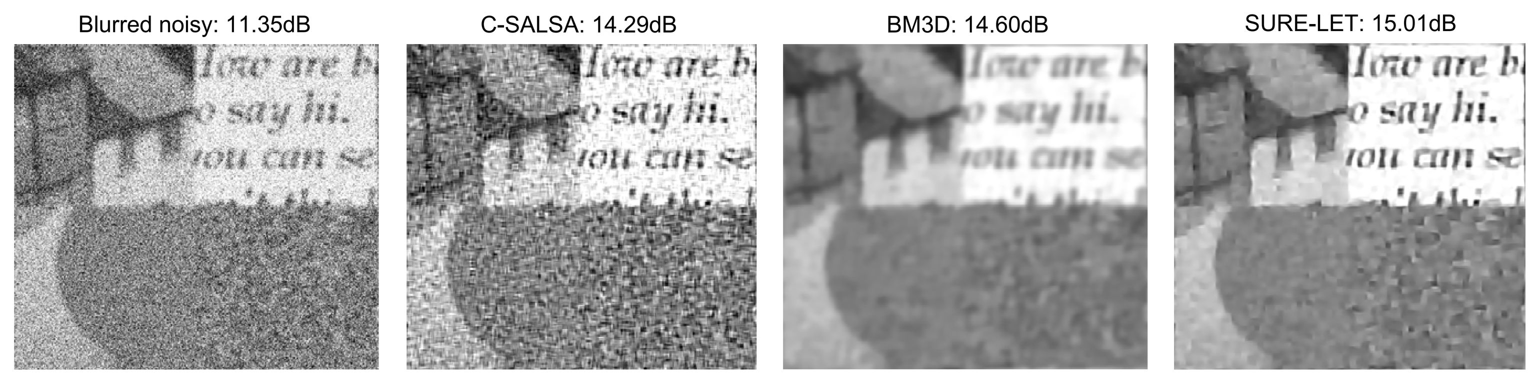


Figure 3: An example: *Mixture* degraded by 9×9 uniform blur with noise std $\sigma = 50$.

Conclusion

- The framework of the proposed SURE-LET approach:
  - extension of **SURE** to deconvolution problem as the objective functional;
  - **linear parametrization** of the processing.
- The originality of the presented work:
  - to use **multiple Wiener filterings** with different but fixed regularization parameters, to avoid empirical adjustment.
- The potential of the presented work:
  - **great flexibility**: take advantage of all the degrees of freedom in the design of the elementary function $\mathbf{F}_k$;
  - **limited computational cost**: fast and exact to solve a linear system of equations;
  - **robustness**: to all noise levels.

Reference

- F. Xue, F. Luisier and T. Blu. "Multi-Wiener SURE-LET Deconvolution", IEEE Transactions on Image Processing, Vol. 22 (5), pp. 1954-1968, 2013.