Construction of an Orthonormal Complex Multiresolution Analysis

Liying Wei and Thierry Blu

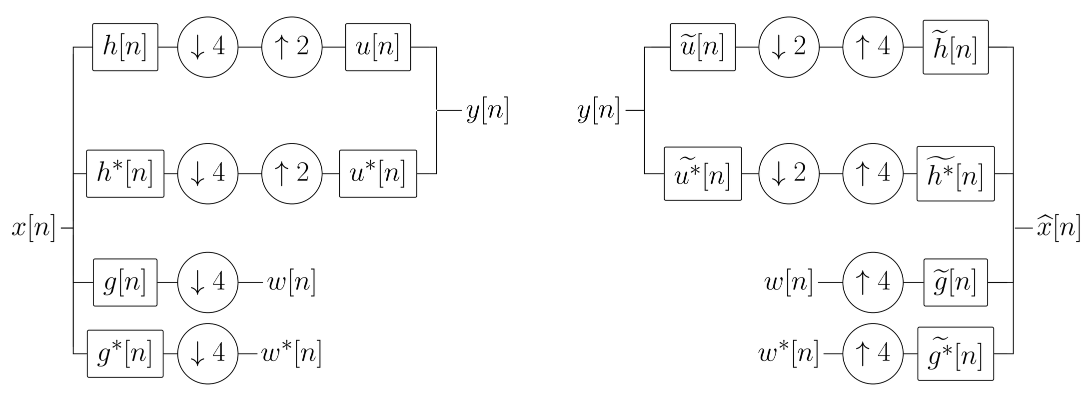
Multimedia & Signal Processing Group, The Chinese University of Hong Kong (CUHK)

香港中文大學 The Chinese University of Hong Kong

Summary

We design two complex filters $\{h[n], g[n]\}$ for the filter bank structure as shown in Fig. 1 based on two atom functions $\{\rho_0^\alpha(t), \rho_{1/2}^\alpha(t)\}$, such that:

- they generate an orthonormal multiwavelet basis;
- the two scaling functions $\{\phi_0(t), \phi_1(t)\}$ are real-valued;
- the two complex conjugate wavelets $\{\psi(t), \psi^*(t)\}$ have their frequency responses supported either on the positive or negative frequencies;
- the resulting complex wavelet transform is non-redundant and able to distinguish $\pm 45°$ diagonal features.



(a) Analysis filter bank (b) Synthesis filter bank

Figure 1: The orthoconjugate filter bank structure for 1D non-redundant complex wavelet transform (NRCWT) implementation. $u[n] = \{\frac{1}{\sqrt{2}}, \frac{j}{\sqrt{2}}\}$.

Complex multiresolution analysis

The Hilbert-pair atom functions

$$\rho_0^\alpha(t) = |t|^\alpha, \quad \rho_{1/2}^\alpha(t) = |t|^\alpha \mathrm{sgn(t)}$$

satisfy a scaling property of the form: $\rho^\alpha(t/2) = 2^{-\alpha}\rho^\alpha(t)$. The multiresolution scaling space V_j is generated by the shifts of these two functions, i.e.,

$$V_j = \mathrm{Span}_{n\in\mathbb{Z}}\{\rho_0^\alpha(2^j t - n), \rho_{1/2}^\alpha(2^j t - n)\}.$$

The scaling space V_j, wavelet spaces W_j and W_j^* satisfy

$$V_j \subset \mathbf{L}^2(\mathbb{R}), \quad V_j \oplus W_j \oplus W_j^* = V_{j+1}, \quad f(t) \in V_j \Longleftrightarrow f(2^{-j}t) \in V_0.$$

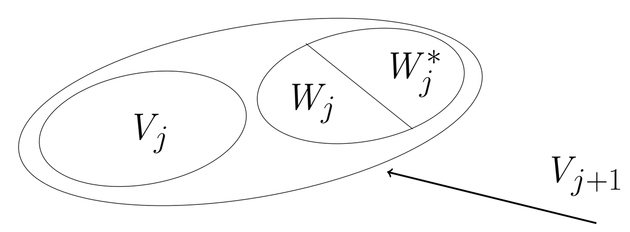


Figure 2: Geometrical structure of complex multiresolution analysis.

Conditions

- The *frequency responses* of the functions in W_j are supported in $[0, +\infty[$, which implies that the frequency responses of the functions in W_j^* are supported in $]-\infty, 0]$,
- Orthogonality: $V_j \perp W_j$ and $V_j \perp W_j^*$,

Result

- There is a *unique* solution for W_j.
- Decomposes $\mathbf{L}^2(\mathbb{R})$ as

$$\mathbf{L}^2(\mathbb{R}) = \overbrace{\ldots \oplus W_{j-1} \oplus W_j \oplus W_{j+1} \oplus \ldots}^{\text{positive frequencies}} \oplus \underbrace{\ldots \oplus W_{j-1}^* \oplus W_j^* \oplus W_{j+1}^* \oplus \ldots}_{\text{negative frequencies}}$$

Problem formulation

Our problem is to find an orthonormal basis for the spaces V_0 and W_0 that satisfy our conditions; i.e., solve for the coefficient matrices $C[n]$ and $D[n]$,

$$\begin{bmatrix}\phi_0(t)\\ \phi_1(t)\end{bmatrix} = \sum_n C[n]\begin{bmatrix}\rho_0^\alpha(t-n)\\ \rho_{1/2}^\alpha(t-n)\end{bmatrix}, \quad \begin{bmatrix}\psi(t)\\ \psi^*(t)\end{bmatrix} = \sum_n D[n]\begin{bmatrix}\rho_0^\alpha(2t-n)\\ \rho_{1/2}^\alpha(2t-n)\end{bmatrix},$$

such that:

- $\phi_0(t)$ and $\phi_1(t)$ are real-valued;
- the associated (complex-valued) wavelets $\psi(t)$ and $\psi^*(t)$ have one-sided frequency support;
- $\phi_0(t)$, $\phi_1(t)$, $\psi(t)$ and $\psi^*(t)$ are jointly orthonormal.

Solution

The frequency responses of $C[n]$ and $D[n]$ are given by

$$\boldsymbol{C}(e^{j\omega}) = \frac{\sqrt{2}}{4}\begin{bmatrix}\frac{\mathrm{sgn}(\omega)\,e^{j\frac{\omega}{2}}}{\sqrt{a(e^{j\omega})}} - \frac{\mathrm{sgn}(\omega)\,e^{j\frac{\omega}{2}}}{\sqrt{a(e^{-j\omega})}} & \frac{j\,\mathrm{sgn}(\omega)\,e^{j\frac{\omega}{2}}}{\sqrt{a(e^{j\omega})}} + \frac{j\,\mathrm{sgn}(\omega)\,e^{j\frac{\omega}{2}}}{\sqrt{a(e^{-j\omega})}}\\ \frac{1}{\sqrt{a(e^{j\omega})}} + \frac{1}{\sqrt{a(e^{-j\omega})}} & \frac{j}{\sqrt{a(e^{j\omega})}} - \frac{j}{\sqrt{a(e^{-j\omega})}}\end{bmatrix},$$

$$\boldsymbol{D}(e^{j\omega}) = e^{-j\omega}\begin{bmatrix}\sqrt{\frac{1}{a(e^{j\omega})} - \frac{2^{-2(\alpha+1)}}{a(e^{j2\omega})}} & j\sqrt{\frac{1}{a(e^{j\omega})} - \frac{2^{-2(\alpha+1)}}{a(e^{j2\omega})}}\\ \sqrt{\frac{1}{a(e^{-j\omega})} - \frac{2^{-2(\alpha+1)}}{a(e^{-j2\omega})}} & -j\sqrt{\frac{1}{a(e^{-j\omega})} - \frac{2^{-2(\alpha+1)}}{a(e^{-j2\omega})}}\end{bmatrix},$$

where $a(e^{j\omega}) = \sum_k \frac{1+\mathrm{sgn}(\omega+2k\pi)}{|\omega+2k\pi|^{2(\alpha+1)}}$.

The corresponding scaling functions and wavelets can be expressed as

$$\widehat{\phi}_0(\omega) = \frac{\sqrt{2}}{4}\left[b(\omega) + b(-\omega)\right] e^{j\frac{\omega}{2}}, \quad \widehat{\phi}_1(\omega) = \frac{\sqrt{2}}{4}\left[b(\omega) + b(-\omega)\right], \qquad (1a)$$

$$\widehat{\psi}(\omega) = \frac{\sqrt{2}}{2}\sqrt{b^2\left(\frac{\omega}{2}\right) - b^2(\omega)}\, e^{-j\frac{\omega}{2}}, \qquad (1b)$$

where $b(\omega) = \frac{1}{\sqrt{a(e^{j\omega})}}\,\frac{1+\mathrm{sgn}(\omega)}{|\omega|^{\alpha+1}}$ $\rightsquigarrow$

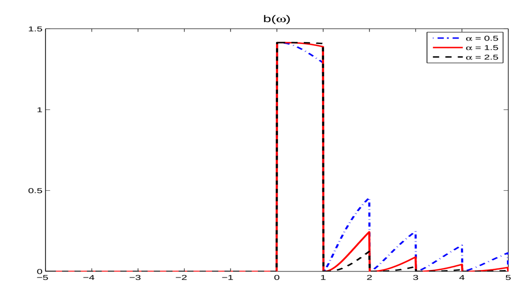
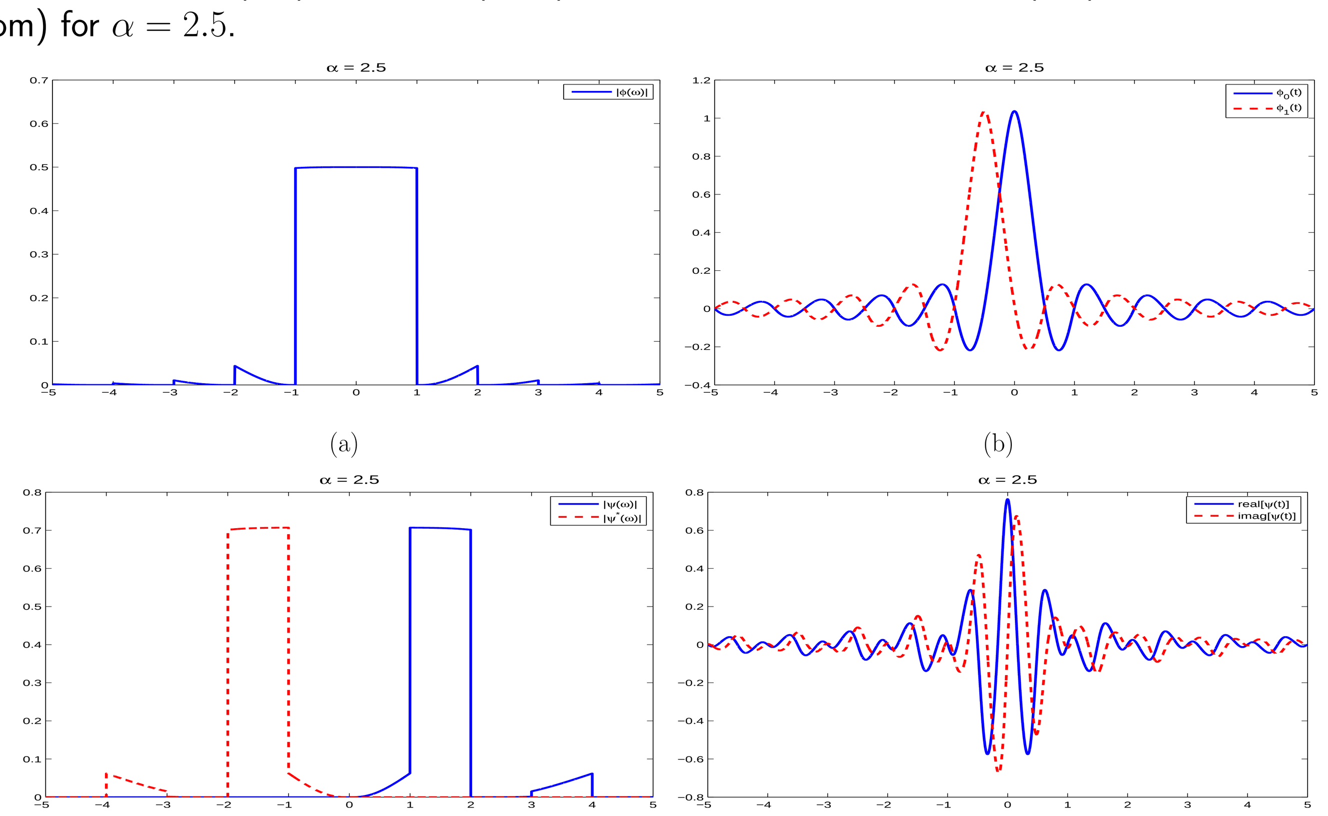


Figure 3: Frequency (left) and time (right) representation of the scaling (top) and wavelet functions (bottom) for $\alpha = 2.5$.



(a) (b) (c) (d)

The frequency responses of $\{h[n], g[n]\}$ are given by

$$H(e^{j\omega}) = \frac{\mathrm{sgn}[\cos(\omega)]\,e^{j\omega} - j\,e^{-j\omega}}{2^{\alpha+1}}\sqrt{\frac{a(e^{j2|\omega|})}{a(e^{j4|\omega|})}}, \qquad (2a)$$

$$G(e^{j\omega}) = \{1 + \mathrm{sgn}[\sin(\omega)]\}\,e^{-j3\omega}\sqrt{1 - 2^{-2(\alpha+1)}\frac{a(e^{j2\omega})}{a(e^{j4\omega})}}. \qquad (2b)$$

Figure 4: Magnitude of the complex filters $\{H(e^{j\omega}), G(e^{j\omega})\}$.

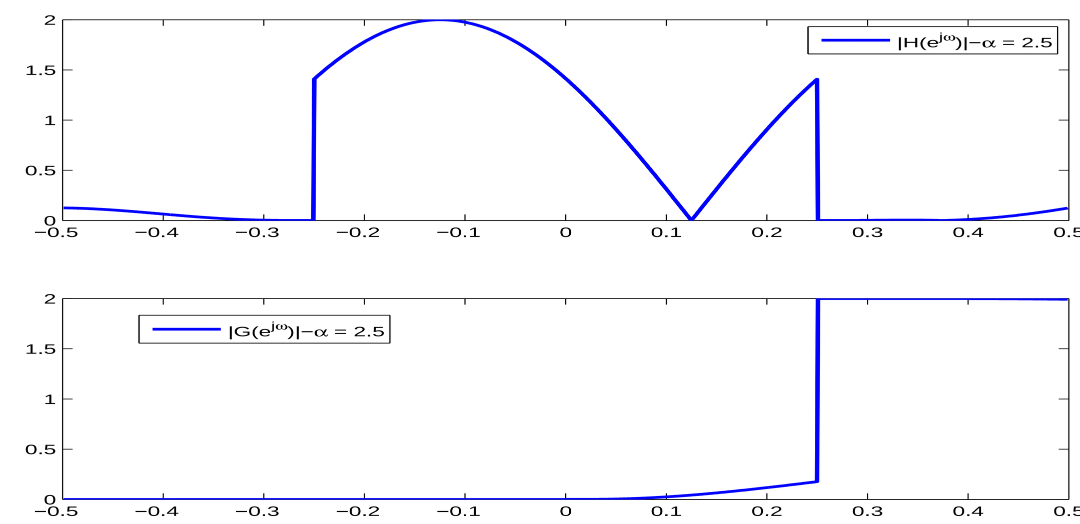


Figure 5: Frequency-domain energy localization of the orthonormal multiwavelet basis.

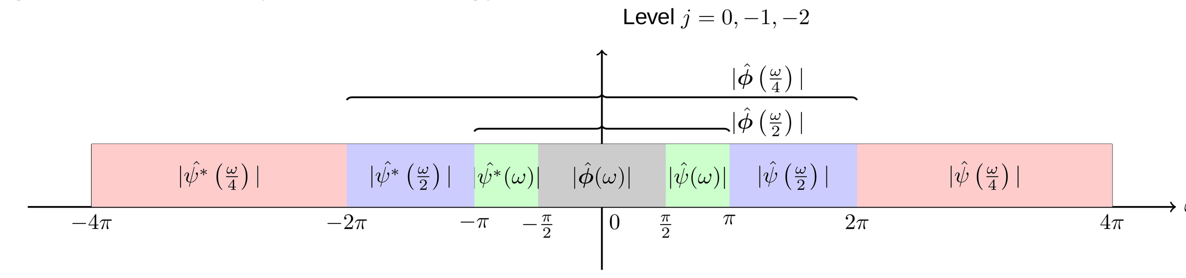


Figure 6: Frequency-domain energy localization of the orthonormal multiwavelet basis.

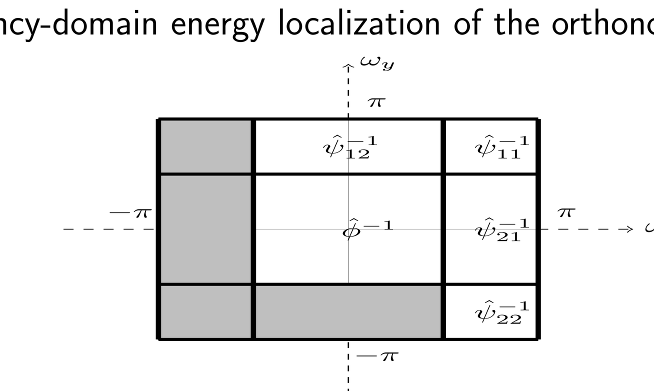
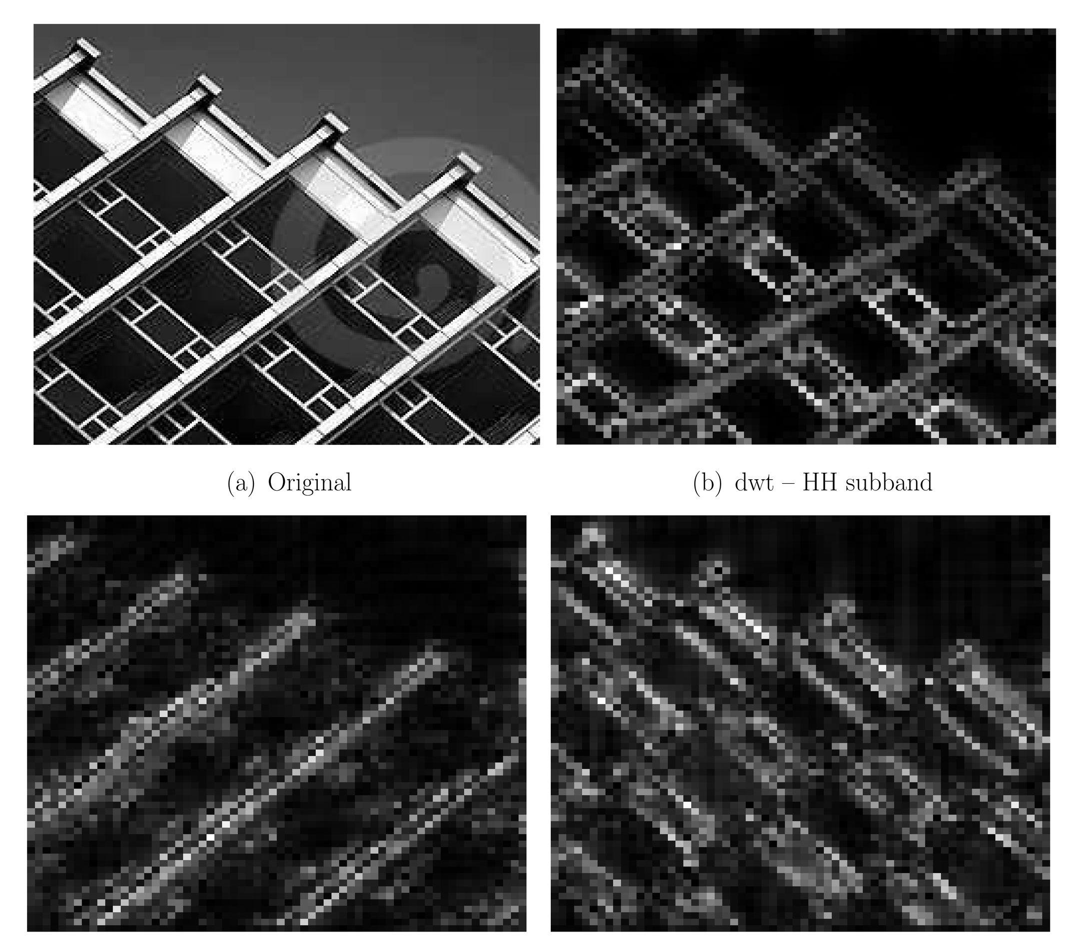


Figure 7: Demonstration of directivities between proposed 2D NRCWT and traditional 2D DWT. The decomposition level is 1. The DWT used is the fractional ($\alpha = 4.5, \tau = 0$) orthonormal B-spline.



(a) Original (b) dwt – HH subband

(c) cwt – '+45° diagonal' ($\hat{\psi}_{11}^{-1}$) subband (d) cwt – '−45° diagonal' ($\hat{\psi}_{22}^{-1}$) subband