Sampling Piecewise Sinusoidal Signals With Finite Rate of Innovation Methods

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***Abstract*—We consider the problem of sampling piecewise sinusoidal signals. Classical sampling theory does not enable perfect reconstruction of such signals since they are not band-limited. However, they can be characterized by a finite number of parameters, namely, the frequency, amplitude, and phase of the sinusoids and the location of the discontinuities. In this paper, we show that under certain hypotheses on the sampling kernel, it is possible to perfectly recover the parameters that define the piecewise sinusoidal signal from its sampled version. In particular, we show that, at least theoretically, it is possible to recover piecewise sine waves with arbitrarily high frequencies and arbitrarily close switching points. Extensions of the method are also presented such as the recovery of combinations of piecewise sine waves and polynomials. Finally, we study the effect of noise and present a robust reconstruction algorithm that is stable down to SNR levels of 7 [dB].**

***Index Terms*—Annihilating filter method, piecewise sinusoidal signals, sampling methods, spline functions.**

I. Introduction

MOST digital acquisition systems involve the conversion of signals from analog to digital. Usually, the device is modeled with a smoothing kernel $\varphi(t)$ and a uniform sampling period $T > 0$. Following this setup, the observed discrete-time signal is given by

$$y[k] = \int_{-\infty}^{\infty} \varphi\left(\frac{t}{T} - k\right) x(t) dt = \left\langle \varphi\left(\frac{t}{T} - k\right), x(t) \right\rangle \quad (1)$$

with $k \in \mathbb{Z}$ as shown in Fig. 1. The fundamental problem of sampling is to recover the original continuous-time waveform $x(t)$ using the set of samples $y[k]$. In the case where the signal is band-limited, the answer due to Shannon is well known [2]. The theorem states that the signal is completely determined by its samples given that the sampling rate $f_s = 1/T$ is greater than or equal to twice the highest frequency component of $x(t)$. The original signal is recovered with $x(t) = \sum_{k\in\mathbb{Z}} y[k]\mathrm{sinc}(t/T - k)$, where $\mathrm{sinc}(t) = \sin(\pi t)/\pi t$ and $y[k] = x(kT)$. The problems arise when the band of $x(t)$ is unlimited—for instance, due to a discontinuity. From a Shannon point of view, these events are seen as infinite innovation processes and therefore require an infinite number of samples. Hence, events concentrated in time are not precisely measurable.

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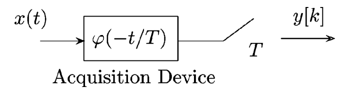


Fig. 1. Sampling setup. The continuous-time signal $x(t)$ is filtered by the acquisition device and sampled with period T. The observed samples are $y[k] = \langle \varphi(t/T - k), x(t) \rangle$.

A sampling scheme has recently been developed by Vetterli *et al.* [3], where it is made possible to sample and perfectly reconstruct signals that are not band-limited but are completely determined by a finite number of parameters. Such signals are said to have a finite rate of innovation (FRI). For instance, the authors derive a method to recover some classes of FRI signals such as streams of Diracs, differentiated Diracs, and piecewise polynomials using sinc or Gaussian kernels. Later, in [4] and [5], it was shown that these signals can also be recovered using more realistic compact support sampling kernels such as those satisfying the Strang–Fix conditions [6], exponential splines [7] and functions with a rational Fourier transform. The case of nonuniform samples across multiple channels has been studied in [8]. The reconstruction process for these schemes is based on the annihilating filter method, a tool widely used in spectral estimation [9], error correction coding [10], interpolation [11], and solving inverse problems [12]–[15]. These results provide an answer for precise time localization (i.e., Diracs and polynomial signals) but in some sense lack frequency localization capabilities.

In this paper, we extend FRI theory to oscillating functions. In particular, we investigate the case where the continuous-time signal is piecewise sinusoidal therefore it contains both time and frequency components. More precisely, we consider signals of the type

$$x(t) = \sum_{d=1}^{D} \sum_{n=1}^{N} A_{d,n} \cos(\omega_{d,n} t + \phi_{d,n}) \xi_d(t) \quad (2)$$

where $\omega_{d,n}$, $A_{d,n}$, and $\phi_{d,n}$ are constant parameters and

$$\xi_d(t) = u(t - t_d) - u(t - t_{d+1}),$$
$$0 \leq t_1 < \cdots t_d < t_{d+1} \cdots < t_{D+1} \leq \infty$$

where $u(t)$ is the Heaviside step function; and study their reconstruction from the samples $y[k]$ given in (1). Such signals are notoriously difficult to reconstruct since they are sparse neither in time nor in frequency. For this reason, the schemes in [3]–[5] as well as the Shannon-type schemes would not enable an exact recovery. However, such signals have a finite rate of innovation, and we demonstrate that it is possible to retrieve the parameters $\omega_{d,n}$, $A_{d,n}$ and $\phi_{d,n}$ of the sinusoids along with the exact locations t_d given certain conditions on the sampling kernel $\varphi(t)$. Note that similar cases have been studied in the FRI context. For example, in [16], the authors deal with band-limited signals that are corrupted by additive shot noise (i.e., Diracs). The case of band-limited signals added to piecewise polynomial signals was also considered in [17]. These types of signals, however, do not encompass the piecewise sinusoidal signal defined in (2).

It is also worth mentioning that a lot of attention has recently been given to the problem of recovering sparse signals from a nonuniform set of samples [18], [19]. These works deal with discrete signals that have a sparse representation in a basis or frame. Extensions to the case of analog signals belonging to a union of shift-invariant subspaces were considered in [20]–[22]. The signals of interest in this paper, however, are not sparse in a basis or frame nor lie in a union of shift-invariant subspaces but have a sparse parametric representation. That is, they can be represented with a finite number of parameters per unit of time.

This paper derives two methods to retrieve exactly continuous-time piecewise sinusoidal signals from their sampled version. Sections II and III discuss the sampling kernels that can be used in our scheme and recall some of the aspects of annihilating filter theory. Using these kernels, Section IV derives a global method for retrieving the parameters of a general piecewise sinusoidal signal. Section V discusses local reconstruction methods that have a lower complexity. In Section VI, we briefly discuss some extensions of the algorithm, namely, adding piecewise polynomials to piecewise sine waves. Section VII deals with noisy observations and presents a robust algorithm for sampling piecewise sinusoidal signals in the presence of noise. We conclude in Section VIII.

II. Sampling Kernels

Many sampling schemes such as the classical Shannon reconstruction [2] and some of the original FRI schemes [3] rely on the ideal low-pass filter (i.e., the sinc function). This filter is not realizable in practice since it is of infinite support. It is therefore attractive to develop sampling schemes where the kernels are physically valid and realizable. It was recently shown that FRI sampling schemes may be used with sampling kernels that are of compact support [4], [5]. In this section, we present these kernels.

A. Polynomial Reproducing Kernels

A polynomial reproducing kernel $\varphi(t)$ is a function that, together with its shifted version, is able to reproduce polynomials. That is, for a given set of values $m = 0, 1, \ldots, M$, it is possible to have

$$\sum_{k\in\mathbb{Z}} c_{m,k}\varphi\left(\frac{t}{T}-k\right) = \left(\frac{t}{T}\right)^m$$

given the right choice of weights $c_{m,k}$. Strang and Fix [6] proved that the necessary and sufficient conditions for a function to have the above property are

$$\hat{\varphi}(0) \neq 0$$
$$\frac{d^m\hat{\varphi}(\omega)}{d\omega^m}\Big|_{\omega=2k\pi} = 0, \quad k \neq 0,\ m = 0, \ldots, M$$

where $\hat{\varphi}(\omega)$ is the Fourier transform of $\varphi(t)$. Perhaps the most basic and intuitive such kernels are the classical B-splines [23]. The B-spline of degree zero is a function with Fourier transform

$$\hat{\beta}_0(\omega) = \frac{1-e^{-j\omega}}{j\omega}.$$

The higher order B-splines of degree N are obtained through $N+1$ successive convolutions of $\beta_0(t)$ such that $\hat{\beta}_N(\omega) = \left((1-e^{-j\omega})/j\omega\right)^{N+1}$, and they are able to reproduce polynomials of degree zero to N. This property follows directly from the Strang–Fix condition above.

B. Exponential Reproducing Kernels

Similarly to the polynomial reproducing kernels, an exponential reproducing kernel $\varphi(t)$ is a function that, together with its shifted version, is able to reproduce exponentials. That is, for any given set of $M+1$ values $(\alpha_0, \ldots, \alpha_M)$, it is possible to have

$$\sum_{k\in\mathbb{Z}} c_{m,k}\varphi\left(\frac{t}{T}-k\right) = e^{\alpha_m t/T}, \quad m = 0, 1, \ldots, M \tag{3}$$

given the right choice of weights $c_{m,k}$. Note that α_m may be complex. One important family of such kernels is the exponential splines (E-splines) that appeared in early works such as [24]–[27] and were further studied in [7]. These functions are extensions of the classical B-splines described above in that they are built with exponential segments instead of polynomial ones. The first-order E-spline is a function $\beta_{\alpha_n}(t)$ with Fourier transform $\hat{\beta}_{\alpha_n}(\omega) = (1-e^{\alpha_n - j\omega})/(j\omega-\alpha_n)$. The E-splines of degree N are constructed by N successive convolutions of first-order ones

$$\hat{\beta}_{\vec{\alpha}}(\omega) = \prod_{n=1}^{N} \frac{1-e^{\alpha_n - j\omega}}{j\omega - \alpha_n} \tag{4}$$

where $\vec{\alpha} = (\alpha_1, \ldots, \alpha_N)$. A series of interesting properties are derived in [7]. In particular, it is shown that an E-spline has compact support and can reproduce any exponential in the subspace spanned by $\{e^{\alpha_1 t}, \ldots, e^{\alpha_N t}\}$. Furthermore, since the exponential reproduction property is preserved through convolution [7], we have that any kernel of the form $\varphi(t) * \beta_{\vec{\alpha}}(t)$ is also able to reproduce the same exponentials as above.

III. Annihilating Filters and Differential Operators

In this section, we recall the notions of annihilating filter and differential operator that are at the heart of the sampling schemes developed in this paper. In particular, we recall the annihilating filter method and show how the annihilating filters in the case of exponential signals are related to the E-splines. We also show how a piecewise exponential signal may be converted into a stream of differentiated Diracs using an appropriate differential operator.

A. The Annihilating Filter Method

Assume that a discrete-time signal $s[k]$ is made of weighted exponentials such that $s[k] = \sum_{n=1}^{N} a_n u_n^k$ with $u_n \in \mathbb{C}$ and assume we wish to retrieve the exponentials u_n and the weights a_n of $s[k]$. The filter $h[k]$ with z-transform

$$H_{\vec{u}}(z) = \prod_{n=1}^{N}(1 - u_n z^{-1}) \tag{5}$$

and $\vec{u} = (u_1, \ldots, u_N)$ is called annihilating filter of $s[k]$ since $(h * s)[k] = 0\ \forall k \in \mathbb{Z}$. We can therefore construct the system of equations:

$$\begin{pmatrix} \vdots & \vdots & \cdots & \vdots \\ s[0] & s[-1] & \cdots & s[-N] \\ s[1] & s[0] & \cdots & s[-N+1] \\ \vdots & \vdots & \ddots & \vdots \\ s[N] & s[N-1] & \cdots & s[0] \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix} \begin{pmatrix} h[0] \\ h[1] \\ \vdots \\ h[N] \end{pmatrix} = 0$$

Notice that $N+1$ equations are sufficient to determine the $h[k]$. Therefore, we write the system in matrix form as

$$\mathbf{S}h = 0 \tag{6}$$

where $\mathbf{S}$ is the appropriate $N+1$ by $N+1$ Toeplitz submatrix involving $2N+1$ samples of $s[k]$. If $s[k]$ admits an annihilatingfilter, we have $Rank(\mathbf{S}) = N$; hence the matrix is rank deficient. The zeros of the filter $H_{\vec{u}}(z)$ uniquely define the u_ns since they are distinct and any filter $h[k]$ satisfying the Toeplitz system in (6) has u_n as its roots. Note that, without loss of generality, we may pose $h[0] = 1$ and solve the system

$$\begin{pmatrix} s[N-1] & s[N-2] & \cdots & s[0] \\ s[N] & s[N-1] & \cdots & s[1] \\ \vdots & \vdots & \ddots & \vdots \\ s[2N-2] & s[2N-1] & \cdots & s[N-1] \end{pmatrix} \begin{pmatrix} h[1] \\ h[2] \\ \vdots \\ h[N] \end{pmatrix} = \begin{pmatrix} -s[N] \\ -s[N+1] \\ \vdots \\ -s[2N-1] \end{pmatrix} \tag{7}$$

which only requires $2N$ samples of $s[k]$. Given the u_ns, the weights a_n are obtained by solving a system of equations using N consecutive samples of $s[k]$. These form the classic Vandermonde system

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ u_1 & u_2 & \cdots & u_N \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{N-1} & u_2^{N-1} & \cdots & u_N^{N-1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} s[0] \\ s[1] \\ \vdots \\ s[N-1] \end{pmatrix}$$

which also has a unique solution given that the u_ns are distinct.

A straightforward extension of the above annihilating filter is that a signal $s[k] = \sum_{n=1}^{N} \sum_{r=0}^{R_n-1} a_{n,r} k^r u_n^k$ is annihilated by the filter

$$H_{\vec{u}}(z) = \prod_{n=1}^{N}(1 - u_n z^{-1})^{R_n} \tag{8}$$

which has multiple roots of order R_n in the u_n. For a more detailed discussion of the annihilating filter method, we refer to [9].

Let us return to the sinusoidal case. Clearly, a filter of the type $H_{\vec{u}}(z)$ will also annihilate a discrete sinusoidal signal $y[k] = \sum_{n=1}^{N} A_n \cos(\omega_n k + \phi_n)$ since it can be written in the form of a linear combination of complex exponentials. In this case, the filter is obtained by posing $\vec{u} = e^{\vec{\alpha}}$ and

$$\vec{\alpha} = (j\omega_1, \ldots, j\omega_N, -j\omega_1, \ldots, -j\omega_N). \tag{9}$$

We simplify the notation by expressing $H_{e^{\vec{\alpha}}}(z)$ as $H_{\vec{\alpha}}(z)$. By comparing (5) with (4) and using $z = e^{j\omega}$, we see that the annihilating filter for a linear combination of exponentials can be expressed with an E-spline as

$$H_{\vec{\alpha}}(e^{j\omega}) = \hat{\beta}_{\vec{\alpha}}(\omega) \prod_{n=1}^{N}(j\omega - \alpha_n) \tag{10}$$

where the second term is a differential operator, which is discussed in the following section.

B. Differential Operators

Let $L\{x(t)\}$ be a differential operator of order N

$$L\{x(t)\} = \frac{d^N x(t)}{dt^N} + a_{N-1}\frac{d^{N-1}x(t)}{dt^{N-1}} + \cdots + a_0 x(t) \tag{11}$$

with constant coefficients $a_n \in \mathbb{C}$. This operator can also be defined by the roots of its characteristic polynomial

$$L(p) = p^N + a_{N-1}p^{N-1} + \cdots + a_0 = \prod_{n=1}^{N}(p - \alpha_n).$$

Using the same notation as in [7], we express the operator as $L_{\vec{\alpha}}$, where $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N)$. Posing $p = j\omega$, we have in the frequency domain

$$L_{\vec{\alpha}}(j\omega) = \prod_{n=1}^{N}(j\omega - \alpha_n).$$

The null space of the operator, denoted $\mathcal{N}_{\vec{\alpha}}$, contains all the solutions to the differential equation $L_{\vec{\alpha}}\{x(t)\} = 0$. It is shown in [7] that $\mathcal{N}_{\vec{\alpha}} = \text{span}\{e^{\alpha_1 t}, \ldots, e^{\alpha_N t}\}$. It is therefore said that the operator has exponential annihilation properties. Moreover, the operator has sinusoidal annihilation properties when $\vec{\alpha}$ is defined as in (9). This follows naturally from the fact that sinusoids are linear combinations of complex exponentials. Therefore, given the right $\vec{\alpha}$, the operator $L_{\vec{\alpha}}$ will produce a zero output for the corresponding sinusoidal input. It is also relevant to mention here that the Green function $g_{\alpha_n}(t)$ of the operator L_{α_n} is a function such that $L_{\alpha_n}\{g_{\alpha_n}(t)\} = \delta(t)$, where $\delta(t)$ is the Dirac distribution. In this case, the Green function is given by $g_{\alpha_n}(t) = e^{\alpha_n t}u(t)$ [7], where $u(t)$ is the Heaviside step function. Consequently, we have that

$$L_{\alpha_n}\{e^{\alpha_n t}u(t - t_n)\} = e^{\alpha_n t}\delta(t - t_n). \tag{12}$$

Finally, by combining (12) with (11), it follows that

$$L_{\vec{\alpha}}\left\{\sum_{n=1}^{N} e^{\alpha_n t}u(t - t_n)\right\} = \sum_{n=1}^{N}\sum_{r=0}^{N-1} w_{n,r}\delta^{(r)}(t - t_n) \tag{13}$$

where $\delta^{(r)}(t)$ is a differentiated Dirac of order r and $w_{n,r}$ are weights that depend on the α_n. Hence, the appropriate differential operator applied to a piecewise exponential signal will produce a stream of differentiated Diracs in the discontinuities t_n.

Note that in [3] and [5], where piecewise polynomial signals are considered, the signal is differentiated with $\alpha_n = 0$. This differentiation of the piecewise polynomial signal leads to a stream of differentiated Diracs that can be retrieved from their samples using signal moments. A similar method can be used in the piecewise sinusoidal case. However, as shown above, the differential operator that produces a stream of differentiated Dirac impulses requires the knowledge of the frequencies of the sine waves [i.e., $\vec{\alpha}$ is as defined in (9)]. Therefore, the methods in [3] and [5] cannot be directly applied.

IV. Reconstruction of Piecewise Sinusoidal Signals Using a Global Approach

All the necessary tools to sample piecewise sinusoidal signals have now been laid down. For mathematical convenience, we write the continuous-time signal as

$$x(t) = \sum_{d=1}^{D}\sum_{n=1}^{2N} A_{d,n} e^{j(\omega_{d,n}t+\phi_{d,n})}\xi_d(t) \tag{14}$$

which is made of D pieces containing a maximum of N sinusoids each. Assume now that this signal is sampled with a kernel $\varphi(t)$ that is able to reproduce exponentials $e^{\alpha_m t}$ with $\alpha_m = \alpha_0 + \lambda m$, where $\alpha_0, \lambda \in \mathbb{C}$ and $m = 0, 1, \ldots, M$. Following previous FRI methods [5], weighting the samples with the appropriate coefficients $c_{m,k}$ gives

$$\begin{aligned}\tau[m] &= \sum_{k\in\mathbb{Z}} c_{m,k} y[k] = \left\langle \sum_{k\in\mathbb{Z}} c_{m,k}\varphi(t-k), x(t)\right\rangle \\ &= \int_{-\infty}^{\infty} e^{\alpha_m t}x(t)dt \end{aligned}\tag{15}$$

where we have used (1) and (3) and set the sampling period to $T = 1$. Note that $\tau[m]$ is an exponential moment of the original continuous-time waveform $x(t)$. In particular, when $\alpha_m = jm\omega_0$, we retrieve the coefficients $\tau[m] = \hat{x}(m\omega_0)$ of the Fourier transform of $x(t)$. Plugging (14) into (15) gives

$$\tau[m] = \sum_{d=1}^{D}\sum_{n=1}^{2N} \bar{A}_{d,n} \frac{\left[e^{t_{d+1}(j\omega_{d,n}+\alpha_m)} - e^{t_d(j\omega_{d,n}+\alpha_m)}\right]}{j\omega_{d,n}+\alpha_m} \tag{16}$$

where $\bar{A}_{d,n} = A_{d,n}e^{j\phi_{d,n}}$. These moments are a sufficient representation of the piecewise sinusoidal signal since the frequencies of the sinusoids and the exact locations of the discontinuities can be found using the annihilating filter method.

Let us define the polynomial $Q(\alpha_m) = \prod_{d=1}^{D}\prod_{n=1}^{2N}(j\omega_{d,n}+\alpha_m)$ of degree $2DN$. Multiplying both sides of (16), we find the expression

$$\begin{aligned}Q(\alpha_m)\tau[m] = \sum_{d=1}^{D}\sum_{n=1}^{2N} &\bar{A}_{d,n}\Phi_{d,n}(\alpha_m) \\ &\times\left[e^{t_{d+1}(j\omega_{d,n}+\alpha_m)} - e^{t_d(j\omega_{d,n}+\alpha_m)}\right]\end{aligned} \tag{17}$$

where $\Phi_{d,n}(\alpha_m)$ is a polynomial of maximum degree $2DN-1$. Recall that we impose $\alpha_m = \alpha_0 + \lambda m$, which means that the right-hand side of (17) is equivalent to $\sum_{d=1}^{D+1}\sum_{r=0}^{2DN-1} b_{r,d}m^r e^{\lambda t_d m}$, where $b_{r,d}$ are weights that depend on α_m but do not need to be computed here. Therefore, a filter of the type

$$H(z) = \prod_{d=1}^{D+1}(1-e^{\lambda t_d}z^{-1})^{2DN} = \sum_{k=0}^{K} h[k]z^{-k}$$

with $K = (D+1)2DN = 2D^2N + 2DN$ will annihilate (17), as shown in (8). It follows that

$$\sum_{k=0}^{K} h[k]Q(\alpha_{n-k})\tau[n-k] = 0 \tag{18}$$

with $n = K, K+1, \ldots, M$. Since Q is a polynomial in α_m, it can be written as

$$Q(\alpha_m) = \sum_{l=0}^{L} r[l]\alpha_m^l$$

where $L = 2DN$. Using this notation, the system in (18) becomes

$$\sum_{k=0}^{K}\sum_{l=0}^{L} h[k]r[l](\alpha_{n-k})^l\tau[n-k] = 0$$

for $n = K, \ldots, M$. For clarity, we write the system in matrix form, which gives (19) as shown at the bottom of the page, where $U = M - K \geq (K+1)(L+1) - 1$. Solving this system with $h[0] = 1$ enables us to find the $r[l]$s. Subsequently,

$$\begin{pmatrix} \tau[K] & \ldots & (\alpha_K)^L\tau[K] & \ldots & \tau[0] & \ldots & (\alpha_0)^L\tau[0] \\ \tau[K+1] & \ldots & (\alpha_{K+1})^L\tau[K+1] & \ldots & \tau[1] & \ldots & (\alpha_1)^L\tau[1] \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \tau[K+U] & \ldots & (\alpha_{K+U})^L\tau[K+U] & \ldots & \tau[U] & \ldots & (\alpha_U)^L\tau[U] \end{pmatrix} \begin{pmatrix} h[0]r[0] \\ \vdots \\ h[0]r[L] \\ \vdots \\ h[K]r[0] \\ \vdots \\ h[K]r[L] \end{pmatrix} = 0 \tag{19}$$

we find the $h[k]$s. The roots of the filter $H(z)$ and the polynomial $Q(\alpha_m)$ give the locations of the switching points[1] and the frequencies of the sine waves, respectively. The number of exponential moments $\tau[m]$ required to build a system with enough equations to find the parameters of the piecewise sinusoidal signal is $M+1 = K+U+1 = 4D^3N^2 + 4D^2N^2 + 4D^2N + 6DN + 1$.

At this point, we have determined all the frequencies of the sinusoids and the locations of the discontinuities. However, the polynomial $Q(\alpha_m)$ does not enable us to distinguish which frequencies are present in which piece. This information, along with the amplitudes and phases of the sinusoids, is found by building a generalized Vandermonde system

$$\tau[m] = \sum_{i=1}^{D+1}\sum_{d=1}^{D}\sum_{n=1}^{2N} \bar{A}_{d,n} \frac{e^{t_i(j\omega_{d,n}+\alpha_m)}}{j\omega_{d,n}+\alpha_m} \tag{20}$$

which requires $2ND(D+1)$ moments $\tau[m]$ and enables us to determine the $\bar{A}_{d,n}$s. This system provides a unique solution since the exponents are distinct. The full algorithm is summarized in Algorithm 1. The derivation above shows that it is possible to reconstruct the piecewise sinusoidal signal in (2) from the set of samples in (1). We therefore have the following result.

Theorem 1: Assume a sampling kernel $\varphi(t)$ that can reproduce exponentials $e^{\alpha_0+\lambda m}$ and $m = 0, 1, \ldots, M$. A piecewise sinusoidal signal with D pieces having a maximum of N sinusoids in each piece is uniquely determined by the samples $y[k] = \langle \varphi(t/T-k), x(t)\rangle$ if $M \geq 4D^3N^2 + 4D^2N^2 + 4D^2N + 6DN$.

Algorithm 1: Global Recovery of a Piecewise Sinusoidal Signal

1: Compute moments $\tau[m]$ in (15).
2: Build the system in (19) and retrieve the annihilating filter $h[k]r[l]$.
3: Set $h[0] = 1$ and retrieve the $r[l]$. Compute the $h[k]$.
4: Compute the roots of the $h[k]$ and $r[l]$ in order to find the t_d and the $\omega_{d,n}$, respectively.
5: Build the system in (20) using the $\tau[m]$ as well as the t_d and $\omega_{d,n}$ computed in the previous step.
6: Retrieve the $\bar{A}_{d,n}$ and compute the $A_{d,n}$ and the $\phi_{d,n}$.

Fig. 2 illustrates the sampling and perfect reconstruction of a truncated sine wave. In this case, $D = 1$ and $N = 1$, and we need to compute exponential moments up to order 18. Note that the method is based on the rate of innovation of the signal only. That is, there are no constraints, for instance, on the frequencies of the sine waves. In particular, we are not limited by the Nyquist frequency. It also means that the locations of the discontinues t_1 and t_2 may be arbitrarily close. In fact, the piecewise sinusoidal signal defined in (2) has a limited number of degrees of freedom since it is zero for $t \notin [t_1, t_{D+1}]$. For this reason, the sampling interval T can, in theory, be arbitrarily large.

[1]Note that in the case where $\lambda = j\omega_0$ is purely imaginary, the ω_0 has to be chosen such that $\omega_0 \leq 2\pi T/t_{D+1}$ in order to avoid ambiguities.

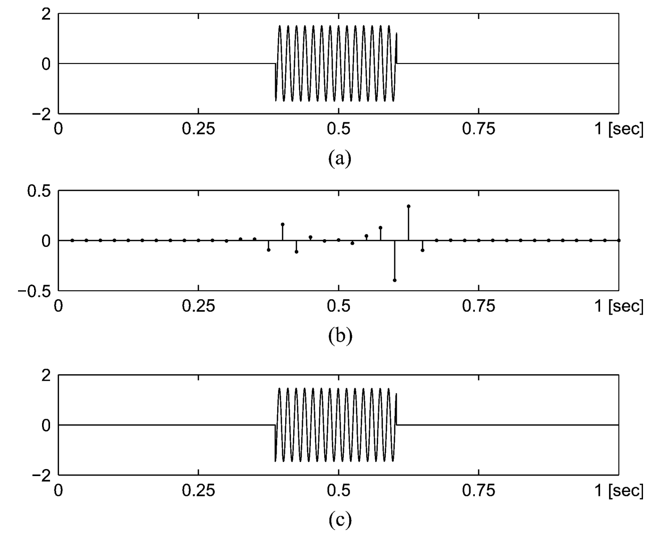


Fig. 2. Sampling a truncated sine wave. (a) The original continuous-time waveform. In this example, we have $t_1 = 0.3877$ [s] and $t_2 = 0.6035$ [s]. The frequency of the sine wave is $\omega_{1,1} = 121\pi$ [rad/s] and the sampling period is $T = 1/40$ [s]. (b) The observed samples. The sampling kernel is an exponential spline with parameters $\alpha_0 = 0$ and $\lambda = j0.082\pi$. (c) The reconstructed signal. Note that the signal is not band-limited, and the frequency of the sine wave itself is higher than the Nyquist rate for the given sampling period.

V. Reconstruction of Piecewise Sinusoidal Signals Using a Local Approach

In the previous section, we saw that it is possible to retrieve the parameters of a sampled piecewise sinusoidal signal given that the sampling kernel is able to reproduce exponentials of a certain degree. This degree, however, increases very rapidly with the number of sinusoids and pieces. In this section, we show that the complexity can be reduced by making further assumptions on the signal and imposing constraints on the sampling period T. These assumptions will allow us to locally reconstruct the signal by retrieving the parameters of two or more consecutive pieces at a time. In the first case, we assume that the frequencies of the sine waves are known and we retrieve the exact locations of the discontinuities. In the second case, we assume that the discontinuities are sufficiently far apart such that a classical spectral estimation method can be run in each piece in order to estimate the frequencies independently of the discontinuities.

A. Local Reconstruction With Known Frequencies

Consider a piecewise sinusoidal signal $x(t)$ as defined in (14) and assume the frequencies $\omega_{d,n}$ are known at the reconstruction. This can be the case, for instance, when information is transmitted using the switching points (or the discontinuities) and we wish to retrieve these locations exactly. The samples $y[k]$ are again given by (1). Since the frequencies of the sine waves are known, we can construct the annihilating filter

$$H_{\vec{\alpha}}(z) = \prod_{d=1}^{D}\prod_{n=1}^{N}(1 - e^{j\omega_{d,n}}z^{-1})(1 - e^{-j\omega_{d,n}}z^{-1})$$

with coefficients $h_{\vec{\alpha}}[k]$ and

$$\vec{\alpha} = (j\omega_{1,1}, \ldots, j\omega_{D,N}, -j\omega_{1,1}, \ldots, -j\omega_{D,N}).$$

Assume now that we apply this filter to the samples $y[k]$. Assuming $T=1$, the expression for the annihilated signal $y'[k]$ gives

$$\begin{aligned} y'[k] &= h_{\vec{\alpha}}[k] * \langle \varphi(t-k), x(t)\rangle \\ &\stackrel{(a)}{=} \frac{1}{2\pi}\langle e^{-j\omega k}\hat{\varphi}(\omega)H_{\vec{\alpha}}(e^{j\omega}), \hat{x}(\omega)\rangle \\ &\stackrel{(b)}{=} \frac{1}{2\pi}\left\langle e^{-j\omega k}\hat{\varphi}(\omega)\hat{\beta}_{\vec{\alpha}}(\omega)L_{\vec{\alpha}}(j\omega), \hat{x}(\omega)\right\rangle \\ &= \langle L_{\vec{\alpha}}\{\varphi(t-k) * \beta_{\vec{\alpha}}(t-k)\}, x(t)\rangle \\ &\stackrel{(c)}{=} \langle \varphi(t-k) * \beta_{\vec{\alpha}}(t-k), L_{\vec{\alpha}}\{x(t)\}\rangle \end{aligned} \tag{21}$$

where (a) follows from Parseval's identity, (b) from (10), and (c) from integration by parts and the fact that $\varphi * \beta_{\vec{\alpha}}$ is of finite support. This means that the coefficients $y'[k]$ represent the samples given by the inner product between a modified $x(t)$ that we call $x'(t) = L_{\vec{\alpha}}\{x(t)\}$ and a new sampling kernel $\varphi'(t) = \varphi(t) * \beta_{\vec{\alpha}}(t)$. Now assume that the sampling kernel $\varphi(t)$ has compact support W. Then the equivalent kernel $\varphi'(t)$ is of compact support $W' = W + 2DN$. Furthermore, according to (13), $x'(t)$ is made only of differentiated Diracs of maximum order $2DN-1$ in the discontinuities. That is, we are left with a signal of the type $x'(t) = \sum_{d=1}^{D+1}\sum_{r=0}^{2DN-1} w_{d,r}\delta^{(r)}(t-t_d)$ for which a sampling theorem exists [4], [5]. Hence, given that the hypotheses of the theorem are met, we are able to perfectly reconstruct $x'(t)$ and retrieve the exact locations t_d. The theorem states that an infinite-length signal made of differentiated Diracs of maximum order $R-1$ can be sampled and perfectly reconstructed given that the sampling kernel is of compact support W, it can reproduce exponentials $e^{\alpha_0+\lambda m}$ or polynomials t^m with $M \geq 2DR-1$, and there are at most D Diracs with DR weights in an interval of length $2DWT$. Since this reproduction capability is preserved through convolution [7], the equivalent kernel $\varphi'(t)$ is able to reproduce the same exponentials or polynomials as $\varphi(t)$. Therefore, $\sum_{k\in\mathbb{Z}} c'_{m,k}\varphi'(t-k) = e^{\alpha_m t}$ or $\sum_{k\in\mathbb{Z}} c'_{m,k}\varphi'(t-k) = t^m$ given the right choice of coefficients $c'_{m,k}$. Hence the classes of kernels used in [4] are also valid in this context.

Similarly to the previous approach, the retrieval of the locations t_d and the weights $w_{d,r}$ of $x'(t)$ is based on the annihilating filter method. As shown in [4] and [5], these parameters can be found using appropriate linear combinations of the samples $y'[k]$. Indeed, using an exponential reproducing kernel, we have the moments

$$\begin{aligned} \tau[m] &= \sum_{k\in\mathbb{Z}} c'_{m,k} y'[k] \\ &= \left\langle \sum_{k\in\mathbb{Z}} c'_{m,k}\varphi'(t-k), x'(t)\right\rangle \\ &= \int_{-\infty}^{\infty} e^{\alpha_m t} x'(t)dt \\ &= \sum_{d=1}^{D+1}\sum_{r=0}^{2DN-1} w_{r,d} m^r e^{\lambda t_d m} \end{aligned} \tag{22}$$

that are made of weighted exponentials. Therefore, a filter of the type $H(z) = \prod_{d=1}^{D+1}(1-e^{\lambda t_d}z^{-1})^{2DN}$ will annihilate $\tau[m]$, and the problem of finding the locations t_d is reduced to that of finding the multiple roots of $H(z)$. This filter can be determined using the Toeplitz matrix in (23) as shown at the bottom of the page, which follows directly from (6). Note that similarly to (7), we may pose $h[0]=1$ and use only $4DN(D+1)-1$ samples of $\tau[m]$. A more detailed description of the location retrieval can be found in [5]. The above discussion is summarized as follows.

Theorem 2: Assume a sampling kernel $\varphi(t)$ that can reproduce exponentials $e^{\alpha_0+\lambda m}$ or polynomials t^m with $m = 0, 1, \ldots, M$ and of compact support W. A piecewise sinusoidal signal with a maximum of N sinusoids in each piece is uniquely determined by the samples $y[k] = \langle \varphi(t/T - k), x(t)\rangle$ if the frequencies $\omega_{d,n}$ are known and there are at most $D+1$ sinusoidal discontinuities in an interval of length $2(D+1)(W+2DN)T$ and $M \geq 4DN(D+1)-1$.

Therefore, given that the sampling kernel satisfies the above properties, a piecewise sinusoidal signal with at most $D+1$ sinusoidal discontinuities in an interval of length ρ can be retrieved if it is sampled at a rate of $T^{-1} \geq 2(D+1)(W+2DN)/\rho$, where N is the maximum number of sinusoids in each piece and W is the support of the sampling kernel $\varphi(t)$.

B. Local Reconstruction With Unknown Frequencies

In the previous section, we saw that the exact locations of the switching points t_d of a piecewise sinusoidal signal can be estimated from its sampled version. The number of moments required in this case was less than in the global method presented in Section IV since, in essence, the estimation of the breakpoints is separated from that of the sine waves.[2] In this section, we show how the local method presented above may be applied even if the frequencies of the sine waves are unknown. The basic idea is to impose that the discontinuities are sufficiently far apart such that a classical spectral estimation method can be run in each piece to estimate the frequencies first.[3]

Assume, for the moment, an original continuous-time signal that is purely sinusoidal with a maximum of N sinusoids. The

[2]For example, in the case where $D=1,\ N=1$, we have $M \geq 7$ instead of $M \geq 18$.

[3]This case was also presented in [1].

$$\begin{pmatrix} \tau[2DN(D+1)] & \ldots & \tau[1] & \tau[0] \\ \tau[2DN(D+1)+1] & \ldots & \tau[2] & \tau[1] \\ \vdots & \ldots & \vdots & \vdots \\ \tau[4DN(D+1)] & \ldots & \tau[2DN(D+1)+1] & \tau[2DN(D+1)] \end{pmatrix} \begin{pmatrix} h[0] \\ h[1] \\ \vdots \\ h[2DN(D+1)] \end{pmatrix} = 0 \tag{23}$$

signal is sampled with a sampling kernel $\varphi(t)$, and the samples are given by

$$y[k]=\sum_{n=1}^{N}\frac{A_n}{2}\left[\hat{\varphi}(-\omega_n)e^{j(\omega_n k+\phi_n)})+\hat{\varphi}(\omega_n)e^{-j(\omega_n k+\phi_n)}\right] \tag{24}$$

From Section III-A, we know that $4N+1$ samples are sufficient to construct the matrix $\mathbf{S}$ in (6) and solve the system of equations in order to determine the annihilating filter $H_{\vec{\alpha}}(z)$. The ω_ns are found using the roots of the filter. Note that as in (7), we may pose $h[0]=1$ and use the fact that the annihilating filter in this case is symmetric. Using these constraints, only $3N$ samples are necessary to determine the annihilating filter. Note that, in this case, the roots of the annihilating filter $H_{\vec{\alpha}}(z)$ are in pairs $z=e^{j\omega_n}$ and $z=e^{-j\omega_n}$. It is therefore necessary to limit the frequencies to $|\omega_n|\leq\pi$ in order to avoid ambiguities. This constraint becomes $|\omega_n|\leq\pi/T$ when the sampling period is not unity. In order to find the amplitudes A_n and the phases ϕ_n, we use $2N$ consecutive samples of $y[k]$ in order to construct a Vandermonde system. For example, in the case where $N=1$, we have the following system:

$$\frac{1}{2}\begin{pmatrix} e^{j\omega_1 k} & e^{-j\omega_1 k}\\ e^{j\omega_1(k+1)} & e^{-j\omega_1(k+1)}\end{pmatrix}\begin{pmatrix} A_1\hat{\varphi}(-\omega_1)e^{j\phi_1}\\ A_1\hat{\varphi}(\omega_1)e^{-j\phi_1}\end{pmatrix}=\begin{pmatrix} y[k]\\ y[k+1]\end{pmatrix}$$

where the unicity of the solution is guarantied since the exponents are distinct. Notice that determining the parameters of the sinusoids is a classical spectral estimation problem [9].

In the piecewise sinusoidal case, the discontinuities influence the samples. Indeed, if the kernel has compact support W, the samples in the interval $[t_d-TW/2, t_d+TW/2]$ are not pure discrete sinusoids as defined in (24). Hence, the sampling period T must be such that there are at least $4N+1$ samples that are not influenced by the discontinuities in each interval $[t_d, t_{d+1}]$. This enables us to use the annihilating filter method to estimate the ω_n. The only apparent difficulty lies in finding the right samples in each piece that are not perturbed by the breakpoints. Recall from Section III-A that the $2N+1$ by $2N+1$ matrix $\mathbf{S}$ admits an annihilating filter when $\text{Rank}(\mathbf{S})=2N$. However, the rank is full when $\mathbf{S}$ is constructed with samples that are influenced by the discontinuities. It follows that the samples that contain purely a sinusoidal contribution can be found by running a window along the k-axis constructing successive matrices and looking at the rank of $\mathbf{S}$. Fig. 3 illustrates the sliding window. In Fig. 3(c), the window contains samples that are influenced by the discontinuity, and the rank of $\mathbf{S}$ is full. However, in Fig. 3(d), the matrix is rank deficient and the annihilating filter method is run to retrieve the parameters of the sinusoids. Once the frequencies have been estimated, the locations of the discontinuities may be found using the method in Section V. Note that in this case, we impose that the discontinuities are sufficiently far apart to retrieve each t_d separately. We therefore have $D=1$. The discussion above is summarized with the following statement.

Theorem 3: Assume a sampling kernel $\varphi(t)$ of compact support W that can reproduce exponentials $e^{\alpha_0+\lambda m}$ or polynomials t^m with $m=0,1,\ldots,M$. A piecewise sinusoidal signal is uniquely determined by the samples $y[k]=\langle\varphi(t/T-k),x(t)\rangle$ if there are at most N sinusoids with maximum absolute frequency $|\omega_{\max}|\leq\pi/T$ in a piece of length $=T(4N+W+1)$ and $M\geq 8N-1$.

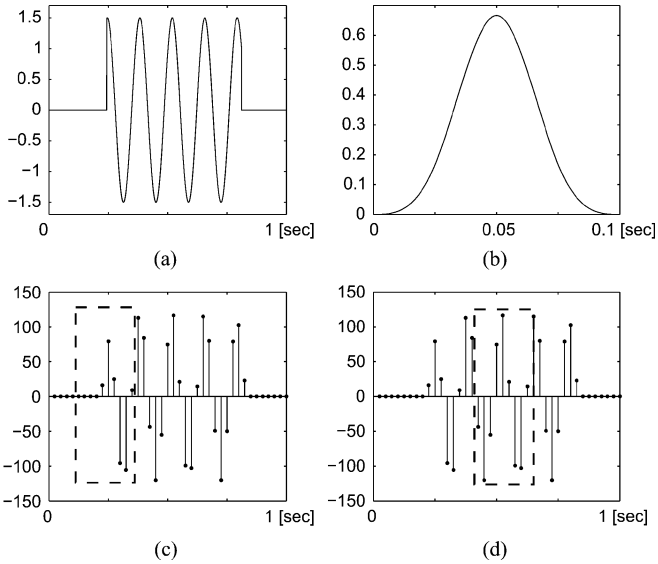


Fig. 3. Determining the sinusoidal part of the pieces. (a) illustrates a truncated sinusoid. Assume, for example, a B-spline sampling kernel $\varphi(t)=\beta_3(t)$ that is of compact support $W=4$ as is depicted in (b). Since the kernel has a certain support, the samples in the vicinity of the discontinuities are not pure discrete sinusoids. Therefore, the rank of matrix $\mathbf{S}$ is full when $\mathbf{S}$ is constructed with the samples in the dashed window depicted in (c). However, $\mathbf{S}$ is rank deficient when the window is chosen as shown (d) since the samples are not influenced by the discontinuities.

Therefore, given that the sampling kernel satisfies the properties of the statement above, a piecewise sinusoidal signal with pieces of minimum length ρ can be retrieved if it is sampled at a rate of $T^{-1}\geq\max\{(4N+W+1)/\rho, |\omega_{\max}|/\pi\}$, where N is the maximum number of sinusoids per piece, $|\omega_{\max}|$ is the maximum absolute frequency of the sinusoids, and W is the support of the sampling kernel $\varphi(t)$.

An overview of the algorithm for the local recovery of piecewise sinusoidal signals is presented in Algorithm 2. A simulation recovering a piecewise sinusoidal signal with three pieces containing one sinusoid per piece is illustrated in Fig. 4. We use a classical B-spline sampling kernel $\beta_7(t)$, as it is capable of reproducing polynomials of maximum degree $8N-1=7$. A numerical simulation for the $N=2$ case is shown in Fig. 5. In both simulations, the reconstructed signal is exact within machine precision.

Algorithm 2: Local Recovery of a Piecewise Sinusoidal Signal

If frequencies of sinusoids are not known:
Run window along k-axis and construct successive $\mathbf{S}$ matrices using the samples $y[k]$.
Find rank-deficient windows $\mathbf{S}\ \forall\ k$.
For each window:
Estimate $(\omega_{d,n}, \phi_{d,n}, A_{d,n})$ on $y[k]$ where $\mathbf{S}$ is rank deficient.
end for
end if
For each pair of consecutive pieces:
Apply annihilating filters for consecutive windows $y'[k]=h_{\vec{\alpha}}*y$.
Compute moments $\tau[m]$ in (22) and build system in (23).
Locations t_d are given by the roots of h.
end for

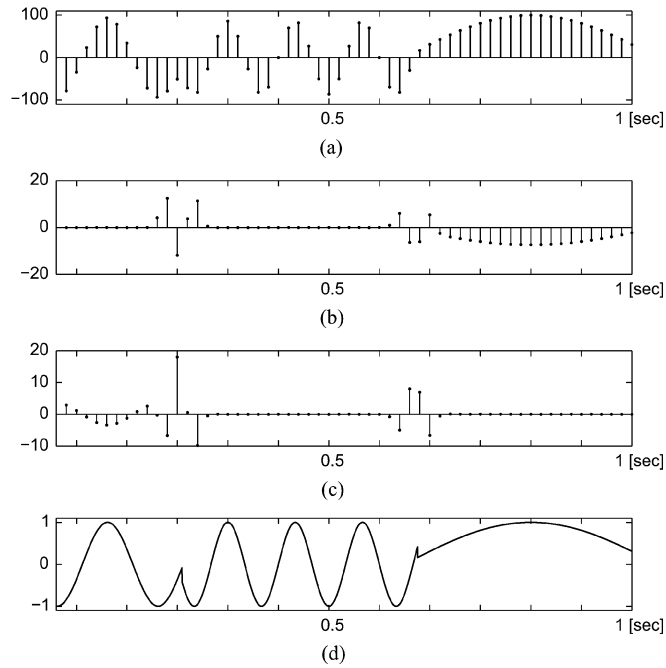


Fig. 4. Sequential recovery of a piecewise sinusoidal signal using the local reconstruction method with unknown frequencies. The observed discrete signal is illustrated in (a). In this example, we have one sine wave per piece and frequencies $\omega_{1,1}$, $\omega_{2,1}$, and $\omega_{3,1}$ in the first, second, and third piece, respectively. The frequencies are determined using the annihilating filter method. The annihilated signal $y'_{t_1}[k] = (y * h_{\vec{\alpha}_1} * h_{\vec{\alpha}_2})[k]$, where $\vec{\alpha}_1 = (j\omega_{1,1}, -j\omega_{1,1})$ and $\vec{\alpha}_2 = (j\omega_{2,1}, -j\omega_{2,1})$, is shown in (b). The nonzero samples in the vicinity of the discontinuity are sufficient to recover the first breakpoint. The second breakpoint can be found by looking at $y'_{t_2}[k] = (y * h_{\vec{\alpha}_2} * h_{\vec{\alpha}_3})[k]$, where $\vec{\alpha}_2 = (j\omega_{2,1}, -j\omega_{2,1})$ and $\vec{\alpha}_3 = (j\omega_{3,1}, -j\omega_{3,1})$, which is depicted in (c). The recovered continuous-time signal is shown in (d).

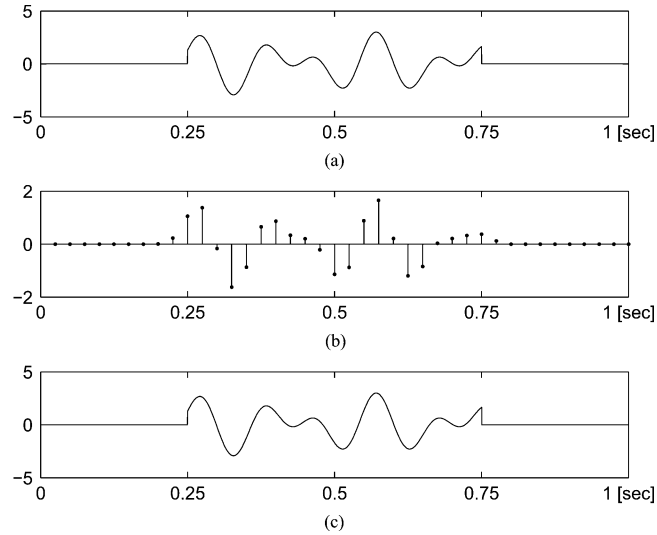


Fig. 5. Numerical simulation of the recovery of a truncated piecewise sinusoidal signal with $N = 2$ sine waves. (a) The continuous-time waveform. (b) The observed samples using the $\beta_7(t)$ sampling kernel. (c) The reconstructed signal.

VI. Joint Recovery of Piecewise Sinusoidal and Polynomial Signals

Sampling piecewise sinusoidal signals using the schemes presented above is based not on the fact that the signals of interest are band-limited but on the fact that they can be represented with a finite number of parameters. It is worth mentioning here that signals that are a combination of piecewise sinusoidal and polynomials pieces are also defined by a finite number of parameters, and they can also be recovered from their sampled versions using the same algorithms. These signals are of the type

$$x(t) = \sum_{d=1}^{D} x_d(t)\xi_d(t)$$

where $x(t) = 0$ for $t < t_1$, $\xi_d(t)$ is as previously defined, and

$$x_d(t) = \sum_{n=1}^{N} A_{d,n} \cos(\omega_{d,n} t + \phi_{d,n}) + \sum_{p=0}^{P-1} B_{d,p} t^p.$$

That is, we have a maximum of N sinusoids and polynomials of maximum degree $P-1$ in each piece. In the following, we will briefly discuss the basic steps to recover the parameters, as they are analogous to the piecewise sinusoidal cases presented in Sections IV and V.

Clearly, the Pth order derivative of $x(t)$ is

$$x^{(P)}(t) = \sum_{d=1}^{D}\sum_{n=1}^{N} \frac{d^P}{dt^P}[A_{d,n}\cos(\omega_{d,n}t + \phi_{d,n})]\xi_d(t) + \sum_{d=1}^{D+1}\sum_{p=0}^{P-1} w_{d,p}\delta^{(p)}(t - t_d)$$

which is a piecewise sinusoidal signal with differentiated Diracs in the discontinuities. Both the global and the local schemes presented above are able to cope with these signals. Therefore, if we are able to relate the observed samples $y[k]$ with the samples $y^{(P)}[k]$ that would have been obtained from $x^{(P)}(t)$, we will be able to recover $x^{(P)}(t)$. The $x(t)$ will then be obtained by integration, which is uniquely defined since we assume that $x(t) = 0$ for $t < t_1$. The relation between the samples $y[k]$ and $y^{(P)}[k]$ is related to B-spline theory and was demonstrated in [5]. Assume we apply the finite difference $y^{(1)}[k] = y[k+1] - y[k]$ to the observed samples. The new set of samples $y^{(1)}[k]$ is equivalent to $y^{(1)}[k] = \langle \varphi(t-k) * \beta_0(t-k), dx(t)/dt \rangle$, where $\beta_0(t)$ is the B-spline of degree zero and where we assume that $T = 1$. Similarly, the Pth-order finite differences lead to the samples

$$y^{(P)}[k] = \left\langle \varphi(t-k) * \beta_{P-1}(t-k), \frac{d^P}{dt^P}x(t) \right\rangle$$

which means the obtained samples are equivalent to those that would have been observed from sampling $x^{(P)}(t)$ with the kernel $\varphi(t) * \beta_{P-1}(t)$. Moreover, since the polynomial and exponential reproduction capability are preserved through convolution, the new kernel is able to reproduce the polynomials or exponentials as well. Hence the sampling schemes presented above are also valid for piecewise sinusoidal and polynomial signals. An example of the sampling of the piecewise polynomial and sinusoidal case is depicted in Fig. 6.

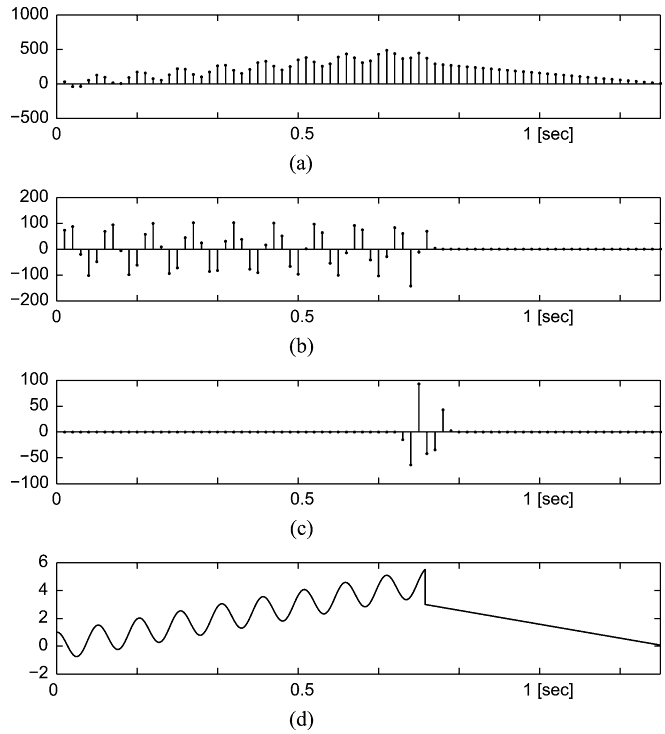


Fig. 6. Sampling a combination of piecewise polynomials and sinusoids. The observed samples are depicted in (a). In the first step, we annihilate the polynomial part by applying the finite difference operator. As shown in (b), we are left only with a piecewise sinusoidal part. The parameters characterizing the sinusoid are retrieved, and the annihilating filter is applied. The samples depicted in (c) contain all the information necessary to find the discontinuity. The recovered continuous-time signal is shown in (d).

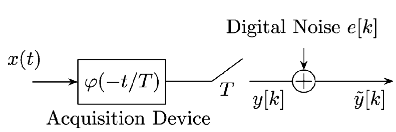


Fig. 7. Sampling setup in the presence of noise. The scenario is identical to the noiseless case. However, we assume that the observed samples $\tilde{y}[k]$ are the noiseless samples $y[k]$ corrupted by digital additive i.i.d. white Gaussian noise $e[k]$.

VII. Dealing With Noise: Problem Setup, Issues, and Solutions

In the first part of this paper, we showed that in the noiseless case, we are able perfectly to reconstruct a continuous-time piecewise sinusoidal signal from its sampled version. In the following, we study the noisy scenario. Reconstruction of finite rate of innovation signals in the presence of noise also has been considered in [28]–[31]. However, these papers concentrate mostly on streams of Diracs and piecewise polynomial signals. Piecewise sinusoidal signals are not considered. In order to study the effect of noise on the estimation of the parameters that define the piecewise sinusoidal signal, we consider additive noise on the samples as illustrated in Fig. 7. Under this model, the observed samples $\tilde{y}[k]$ are given by

$$\tilde{y}[k] = y[k] + e[k] = \int_{-\infty}^{\infty} \varphi(t-k)x(t)dt + e[k] \qquad (25)$$

where we assume that the sampling period T is unity and that the $e[k]$ are independent identically distributed (i.i.d.) and follow a Gaussian distribution with zero mean and variance σ^2.

For clarity, we consider retrieving parameters of a truncated sine wave $x(t) = \cos(\omega_1 t)[u(t-t_1) - u(t-t_2)]$ using the local method with unknown frequency presented in Section V-B. In this context, we are estimating the locations t_1, t_2 and the frequency ω_1 from the observed samples in (25). We will also assume that the locations t_1 and t_2 are sufficiently far apart such that they can be retrieved independently. The sampling kernel $\varphi(t)$ is a polynomial spline $\beta_4(t)$ with Fourier transform $\hat{\beta}_4(\omega) = \left((1-e^{-j\omega})/j\omega\right)^5$. The first step in the reconstruction algorithm consists in estimating the frequency ω_1 of the sine wave. However, we will not delve into this problem here since it is a classical spectral estimation problem, which has been well researched and can be solved using a variety of algorithms. For a detailed view, we refer to [9]. Rather, we will focus on the retrieval of the switching points t_1 and t_2. Note that the samples that contain only the sine information may be located in a similar fashion to that of Section V-B. However, in this case, we compare the largest and the smallest eigenvalues of the successive $\mathbf{S}$ matrices using the samples $\tilde{y}[k]$. The regions where the ratio of the smallest over the largest eigenvalue is smaller than a threshold γ are chosen to estimate the frequency of the sine wave. In order to be more robust to noise, we may choose $N > 1$.

Following the method in Section V-B, we apply the annihilating filter in order to obtain the annihilated samples

$$\tilde{y}'[k] = (y[k] + e[k]) * h_{\vec{\alpha}}[k] = y'[k] + e'[k]$$

and compute the moments $\tau[m]$ with which we recover the switching points. This straightforward application, however, becomes unstable in the presence of noise. The reason for this issue is that the signal-to-noise ratio (SNR[4]) of $\tilde{y}'[k]$, which is used for the estimation of t_1 and t_2, is usually lower than that of $\tilde{y}[k]$. Indeed, by applying the annihilating filter $h_{\vec{\alpha}}[k]$, we are effectively killing most of the power of the signal without reducing the noise.[5] This effect is particularly visible in Fig. 8(b), where the equivalent Dirac $x'(t)$ is buried in the noise. There is therefore a need to design an algorithm to find the locations of t_1 and t_2 that preserves at least some of the energy of the original waveform $x(t)$.

A. A Polyphase Reconstruction Algorithm

In this section, we show that by applying an additional filter to the samples before computing the moments in (22), we are able to improve the estimation of the switching points.

Consider the filter $H_\beta^2(e^{j\omega}) = 1/32(1+e^{-j\omega})^5$ and assume we apply it to the B-spline $\hat{\beta}_4(\omega)$. The resulting function becomes

$$\begin{aligned}\hat{\beta}_4(\omega)H_\beta^2(e^{j\omega}) &= \left(\frac{1-e^{-j\omega}}{j\omega}\right)^5 \frac{1}{32}(1+e^{-j\omega})^5 \\ &= \left(\frac{1-e^{-2j\omega}}{2j\omega}\right)^5 \\ &= \hat{\beta}_4(2\omega) \end{aligned} \qquad (26)$$

[4]We define the SNR as $20\log \|y[k]\|_2/\|e[k]\|_2$, where $\|\cdot\|_2$ is the l_2-norm.

[5]The noise in this case is no longer white but is a filtered noise $e * h_{\vec{\alpha}}$.

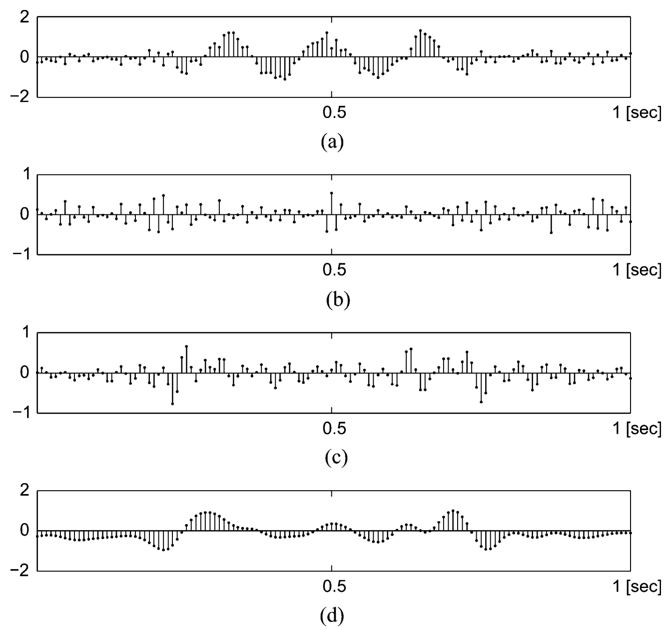


Fig. 8. Applying the annihilating and polyphase filters to noisy samples at SNR= 8 [dB]. (a) The observed noisy samples $\tilde{y}[k]$. (b) The annihilated samples $\tilde{y}'[k] = h_{\vec{\alpha}} * \tilde{y}$. In this case, the equivalent signal $x'(t)$ is buried in noise. (c) The annihilated samples $\tilde{y}''[k] = h_{\beta}^2 * h_{\vec{\alpha}}^2 * h_{\vec{\alpha}} * \tilde{y}$ using the two-phase annihilating filter. (d) The annihilated samples $\tilde{y}^4[k] = h_{\beta}^4 * h_{\vec{\alpha}}^4 * h_{\vec{\alpha}} * \tilde{y}$ using the four-phase annihilating filter. Note that the location of the switching points becomes more distinct when the additional filters are applied.

which is a scaled version of the original B-spline. Similarly, consider the filter $H_{\vec{\alpha}}^2(e^{j\omega}) = 1/4(1+e^{j\omega_1}e^{-j\omega})(1+e^{-j\omega_1}e^{-j\omega})$, where $\vec{\alpha} = (j\omega_1, -j\omega_1)$. This filter together with the original annihilating filter $H_{\vec{\alpha}}(e^{j\omega})$ gives

$$\begin{aligned}
&H_{\vec{\alpha}}(e^{j\omega})H_{\vec{\alpha}}^2(e^{j\omega})\\
&\quad=(1-e^{j\omega_1}e^{-j\omega})(1-e^{-j\omega_1}e^{-j\omega})\\
&\quad\quad\times\frac{1}{4}(1+e^{j\omega_1}e^{-j\omega})(1+e^{-j\omega_1}e^{-j\omega})\\
&\quad=\frac{1}{4}(1-e^{2j\omega_1}e^{-2j\omega})(1-e^{-2j\omega_1}e^{-2j\omega})\\
&\quad=\hat{\beta}_{2\vec{\alpha}}(2\omega)L_{\vec{\alpha}}(j\omega).
\end{aligned} \tag{27}$$

That is, the new annihilating filter can be related to a scaled E-spline and a differential operator. Applying both filters to the samples, we have

$$\begin{aligned}
y''[k] &= h_{\beta}^2 * h_{\vec{\alpha}}^2 * h_{\vec{\alpha}} * \langle \beta_4(t-k), x(t)\rangle\\
&= \frac{1}{2\pi}\langle e^{-j\omega k}\hat{\beta}_4(\omega)H_{\beta}^2(e^{j\omega})H_{\vec{\alpha}}^2(e^{j\omega})H_{\vec{\alpha}}(e^{j\omega}), \hat{x}(\omega)\rangle\\
&= \frac{1}{2\pi}\langle e^{-j\omega k}\hat{\beta}_4(2\omega)\hat{\beta}_{2\vec{\alpha}}(2\omega)L_{\vec{\alpha}}(j\omega), \hat{x}(\omega)\rangle\\
&= \left\langle L_{\vec{\alpha}}\left\{\beta_4\left(\frac{t}{2}-k\right) * \beta_{2\vec{\alpha}}\left(\frac{t}{2}-k\right)\right\}, x(t)\right\rangle\\
&= \left\langle \beta_4\left(\frac{t}{2}-k\right) * \beta_{2\vec{\alpha}}\left(\frac{t}{2}-k\right), L_{\vec{\alpha}}\{x(t)\}\right\rangle
\end{aligned}$$

where we have used the same derivation as in (21) together with (26) and (27). The new noisy samples are given by

$$\tilde{y}''[k] = \left\langle \beta_4\left(\frac{t}{2}-k\right) * \beta_{2\vec{\alpha}}\left(\frac{t}{2}-k\right), L_{\vec{\alpha}}\{x(t)\}\right\rangle + e''[k]$$

where $e''[k] = h_{\beta}^2 * h_{\vec{\alpha}}^2 * h_{\vec{\alpha}} * e$. Thus, these observed samples are equivalent to those that would have been observed if the signal $x'(t) = L_{\vec{\alpha}}\{x(t)\}$ was sampled with the scaled kernel $\varphi''(t) = \beta_4(t/2) * \beta_{2\vec{\alpha}}(t/2)$. Therefore, we may write the samples as

$$\tilde{y}''[k] = \langle \varphi''(t-k), x'(t)\rangle + e''[k].$$

Since the new kernel is an E-spline scaled by a factor of two and T remains the same, we have two times more samples than are necessary to recover t_1. The samples can therefore be decomposed into their polyphase components

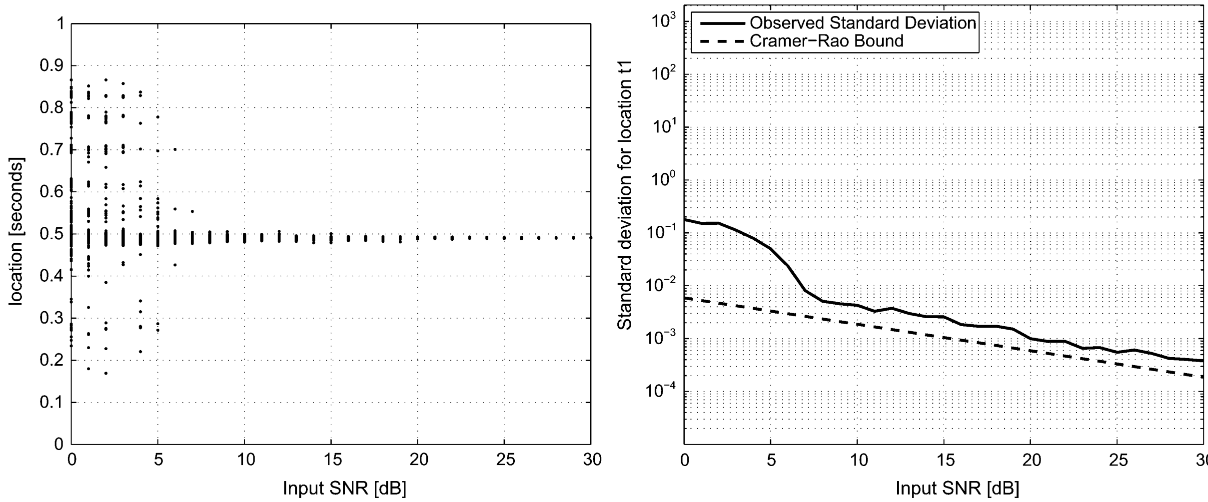
$$\begin{aligned}
\tilde{y}_1''[k] &= \langle \varphi''(t-2k), x'(t)\rangle + e''[2k]\\
\tilde{y}_2''[k] &= \langle \varphi''(t-2k-1), x'(t)\rangle + e''[2k-1].
\end{aligned}$$

Each polyphase component is treated independently, and the corresponding coefficients $c_{m,2k}''$ and $c_{m,2k-1}''$ are obtained using the kernel $\varphi''(t)$. The location t_1 is estimated separately from the even and odd samples, and the final estimate is given by the average of the two obtained locations. This procedure may be iterated in order to create four, eight, or more polyphase components. For instance, the filters in the four-phase case are $H_{\beta}^4(e^{j\omega}) = H_{\beta}^2(e^{j\omega})1/32(1+e^{-2j\omega})^5$ and $H_{\vec{\alpha}}^4(e^{j\omega}) = H_{\vec{\alpha}}^2(e^{j\omega})1/4(1+e^{2j\omega_1}e^{-2j\omega})(1+e^{-2j\omega_1}e^{-2j\omega})$. In the general case of p phases, we denote the samples as $\tilde{y}^p[k] = h_{\beta}^p * h_{\vec{\alpha}}^p * h_{\vec{\alpha}} * \tilde{y}$, and the equivalent sampling kernel is $\varphi^p(t)$ with compact support W_p.

Note that this polyphase reconstruction is somewhat reminiscent of the scenario used in [5] for recovering Diracs in noise. However, while in that paper the polyphase components are obtained through oversampling, the method presented here does not require increase of the sampling rate.

B. Further Denoising With Hard Thresholding

The sampling kernels assumed in the context of this paper are of finite support (i.e., B-splines or E-splines). We may therefore use this property to reduce the noise in the samples. Indeed, if we assume that the discontinuities are sufficiently far apart, we expect to have only W_p nonzero samples of $y^p[k]$, where W_p is the support of the equivalent sampling kernel $\varphi^p(t)$. This is due to the fact that the equivalent signal $x'(t)$ is a sum of differentiated Diracs. Our hard thresholding approach therefore consists in obtaining an initial estimate k_dT of the switching location and setting to zero all the nonzero samples that are not in the interval $kT \in [k_dT - TW_p/2, k_dT + TW_p/2]$ since they are assumed to be purely generated by the additive noise. Assuming the intervals in between switching points have been determined, we obtain the initial estimate of the location k_d by averaging the locations of the maximum and minimum values of the annihilated signal $\tilde{y}^p[k]$. An example of the annihilated signals for different numbers of phases is depicted in Fig. 8(b)–(d). The overall algorithm is described in Algorithm 3.



(a) (b)

Fig. 9. Retrieval of the switching point of a step sine ($\omega_1 = 12.23\pi$ [rad/s] and $t_1 = 0.4907$ [s]) in 128 noisy samples. (a) Scatter plot of the estimated location. (b) Standard deviation (averages over 1000 iterations) of the location retrieval compared to the Cramér–Rao bound.

Algorithm 3: Recovery of a Piecewise Sinusoidal Signal in Noise

1: Run window along k-axis and construct successive $\mathbf{S}$ matrices using the samples $\tilde{y}[k]$.

2: Perform singular value decomposition of $\mathbf{S}\ \forall\ k$.

3: Estimate $(\omega_{d,n}, \phi_{d,n}, A_{d,n})$ on $\tilde{y}[k]$ where smallest eigenvalue over biggest eigenvalue of $\mathbf{S}$ is smaller than threshold γ.

4: **For each** pair of consecutive pieces:

5: Apply annihilating filters $\tilde{y}^p[k] = h_\beta^p * h_{\vec{\alpha}}^p * h_{\vec{\alpha}} * \tilde{y}$ where p represents the number of phases.

6: Estimate switching point k_d using the average of $k_{\max} = \mathrm{argmax}_k\{\tilde{y}^p[k]\}$ and $k_{\min} = \mathrm{argmin}_k\{\tilde{y}^p[k]\}$.

7: Keep only the samples $kT \in [k_dT - TW_p/2, k_dT + TW_p/2]$.

8: **For each** polyphase component:

9: Compute moments $\tau[m]$ in (22) and build system in (23).

10: Locations t_d are given by the roots of h.

11: **end for**

12: Average locations obtained from each polyphase component.

13: **end for**

C. Performance Evaluation

It is of interest here to evaluate the performance of the reconstruction algorithm in the presence of different noise levels. Therefore, we consider the Cramér–Rao bound that provides an answer to the best possible performance of an unbiased estimator. The derivation of the Cramér–Rao bound in the case of additive white Gaussian noise is presented in Appendix A. In this experiment, the acquisition device observes 128 noisy samples with $T = 1/128$ [s] of a truncated sine wave with frequency $\omega_1 = 12.23\pi$ [rad/s] and switching points $t_1 = 0.4907$ [s] and $t_2 = 1$ [s]. Since we assume that the switching points are sufficiently far apart, their locations can be estimated independently. We therefore show the results only for the first discontinuity. The frequency of the sine wave is estimated using Matlab's root-music function, and the location of the switching point is estimated using a four-phase approach and additional hard thresholding. Note that we have also experimented with other frequency estimation methods as well as using the ground truth frequency. Similar results are obtained in all cases. Fig. 9(a) shows the scatter plot for the reconstruction of the switching point t_1 for different SNR levels. The standard deviation of the error (averages over 1000 iterations) of the location retrieval is shown in Fig. 9(b). These simulations show that the proposed reconstruction algorithm behaves well down to noise levels of about 7 [dB]. Fig. 10 illustrates an example of the recovery of a continuous-time piecewise sinusoidal signal (with $t_1 = 0.2441$ [s], $t_2 = 0.7324$ [s], and $\omega_1 = 12.23\pi$ [rad/s]) given 128 noisy samples at an SNR of 8 [dB]. Note that despite the small error in the estimation of the frequency of the sine wave, the estimation of the switching points is accurate.

VIII. Conclusion

We have set out to show that piecewise sinusoids belong to the family of signals with finite rate of innovation and can be sampled and perfectly reconstructed using sampling kernels that reproduce exponentials or polynomials. These classes of kernels are physically realizable and are of compact support. Moreover, combinations of piecewise sinusoids and polynomials also have a finite rate of innovation and can be dealt with using similar sampling schemes.

Since the sampling scheme is limited by the rate of innovation rather than the actual frequency of the continuous-time signal, we are, in theory, capable of retrieving piecewise sine waves with an arbitrarily high frequency along with the exact location of the switching points. We believe, therefore, that the sampling

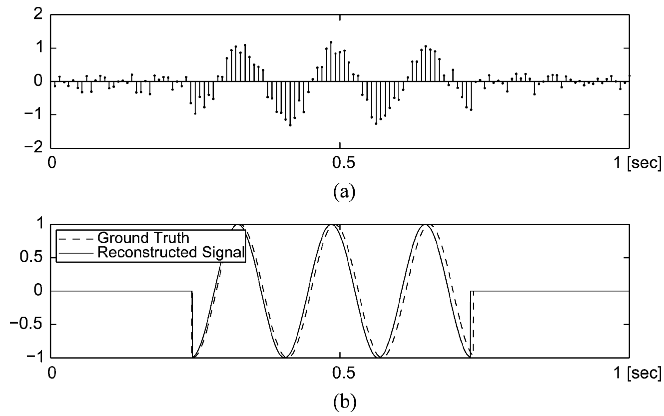


Fig. 10. Recovery of a truncated sine wave at SNR$=8$ [dB]. (a) The observed noisy samples. (b) The reconstructed signal along with the ground truth signal (dashed).

scheme presented may find applications, for example, in spread-spectrum and wide-band communications.

Finally, we studied the effect of noise on the performance of the estimation of the switching points. In doing so, we derived a polyphase reconstruction algorithm that, together with hard thresholding, behaves well with respect to the Cramér–Rao bounds down to SNRs of 7 [dB].

APPENDIX
DERIVATION OF THE CRAMÉR–RAO BOUNDS

The piecewise sinusoidal signal we consider in (2) is defined by the parameter vector

$$\mathbf{\Theta} = (t_1, A_{1,1}, \omega_{1,1}, \phi_{1,1}, \ldots, t_D, A_{D,N}, \omega_{D,N}, \phi_{D,N}, t_{D+1})$$

These parameters are estimated using the observed samples

$$\tilde{y}[k] = \int_{-\infty}^{\infty} \varphi\left(\frac{t}{T} - k\right) x(t) dt + e[k]$$
$$k = 0, 1, \ldots, K$$

where $e[k]$ is i.i.d. additive white Gaussian noise with zero mean and variance σ^2. For clarity, we denote $\tilde{y}[k]$ as a function of the parameter vector with

$$\tilde{y}[k] = f(\mathbf{\Theta}, k) + e[k].$$

The performance of any unbiased estimator $\hat{\mathbf{\Theta}}$ is lower bounded by the Cramér–Rao bound var$(\hat{\mathbf{\Theta}}) \geq I^{-1}(\mathbf{\Theta})$, where $I(\mathbf{\Theta})$ is the Fisher information matrix defined as $I(\mathbf{\Theta}) = E(\nabla l(\mathbf{\Theta}) \nabla l(\mathbf{\Theta})^T)$ and $l(\mathbf{\Theta})$ is the log-likelihood function of the data, i.e., a function of $\mathbf{\Theta}$ conditioned on the measured samples $\tilde{y}[k]$. Recall that the noise $e[k]$ follows the Gaussian distribution

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right).$$

Therefore, we have

$$p(\tilde{y}[k] | \mathbf{\Theta}) = p(\tilde{y}[k] - f(\mathbf{\Theta}, k)).$$

Hence the log-likelihood function is given by

$$\begin{aligned} l(\mathbf{\Theta}) &= \ln p(\tilde{y}[0], \ldots, \tilde{y}[K] | \mathbf{\Theta}) \\ &= \ln \prod_{k=0}^{K} p(\tilde{y}[k] | \mathbf{\Theta}) = \sum_{k=0}^{K} \ln p(\tilde{y}[k] - f(\mathbf{\Theta}, k)) \end{aligned}$$

where we have used the independence of the noise samples $e[k]$. The partial derivative of the log-likelihood function is given by

$$\frac{\partial l}{\partial \Theta_i} = \frac{1}{\sigma^2} \sum_{k=0}^{K} e[k] \frac{\partial f(\mathbf{\Theta}, k)}{\partial \Theta_i}$$
$$\nabla l(\mathbf{\Theta}) = \frac{1}{\sigma^2} \sum_{k=0}^{K} e[k] \nabla f(\mathbf{\Theta}, k)$$

where the ∇ operator denotes the gradient of $f(\mathbf{\Theta}, k)$ with respect to the signal parameters Θ_i. Finally, we determine the Fisher information matrix as

$$\begin{aligned} I(\mathbf{\Theta}) &= E(\nabla l(\mathbf{\Theta}) \nabla l(\mathbf{\Theta})^T) \\ &= E\left(\frac{1}{\sigma^4} \sum_{k=0}^{K} \sum_{m=0}^{M} e[k] e[m] \nabla f(\mathbf{\Theta}, k) \nabla f(\mathbf{\Theta}, m)^T\right) \\ &= \frac{1}{\sigma^4} \sum_{k=0}^{K} \sum_{m=0}^{M} E(e[k] e[m]) \nabla f(\mathbf{\Theta}, k) \nabla f(\mathbf{\Theta}, m)^T \\ &= \frac{1}{\sigma^2} \sum_{k=0}^{K} \nabla f(\mathbf{\Theta}, k) \nabla f(\mathbf{\Theta}, k)^T \end{aligned}$$

where we have used the fact that the noise is independent (i.e., uncorrelated). The Cramér–Rao bound is thus given by

$$\mathrm{CRB}(\Theta) = \sigma^2 \left(\sum_{k=0}^{K} \nabla f(\mathbf{\Theta}, k) \nabla f(\mathbf{\Theta}, k)^T\right)^{-1} \tag{28}$$

with $\nabla f(\mathbf{\Theta}, k) = [\partial f / \partial \Theta_1, \ldots, \partial f / \partial \Theta_{3ND+D+1}]$.

These bounds are complicated to compute for a general piecewise sinusoidal signal. However, we can look at the simpler case where we assume that the observed signal is a single truncated sine wave with known amplitude $A_1 = 1$ and phase $\phi_1 = 0$. The signal is therefore given by

$$x(t) = \cos(\omega_1 t) \xi_1(t),$$
$$\xi_1(t) = u(t - t_1) - u(t - t_2)$$

which is characterized by three parameters: $\mathbf{\Theta} = (t_1, \omega_1, t_2)$. Computing the partial derivatives gives

$$\begin{aligned} \frac{\partial f(\mathbf{\Theta}, k)}{\partial t_1} &= \frac{\partial}{\partial t_1} \int_{-\infty}^{\infty} \cos(\omega_1 t) \xi_1(t) \varphi(t - k) dt \\ &= -\cos(\omega_1 t_1) \varphi(t_1 - k), \\ \frac{\partial f(\mathbf{\Theta}, k)}{\partial t_2} &= \frac{\partial}{\partial t_2} \int_{-\infty}^{\infty} \cos(\omega_1 t) \xi_1(t) \varphi(t - k) dt \\ &= \cos(\omega_1 t_2) \varphi(t_2 - k), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial f(\mathbf{\Theta}, k)}{\partial \omega_1} &= \frac{\partial}{\partial \omega_1} \int_{-\infty}^{\infty} \cos(\omega_1 t) \xi_1(t) \varphi(t - k) dt \\ &= -\int_{t_1}^{t_2} t \sin(\omega_1 t) \varphi(t - k) dt. \end{aligned}$$

Using these three relations enables to evaluate numerically the Cramér–Rao bound in (28).

REFERENCES

[1] J. Berent and P. L. Dragotti, "Perfect reconstruction schemes for sampling piecewise sinusoidal signals," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process. (ICASSP)*, May 2006, pp. 377–380.

[2] C. Shannon, "Communications in the presence of noise," *Proc. IRE*, vol. 37, pp. 10–21, Jan. 1949.

[3] M. Vetterli, P. Marziliano, and T. Blu, "Sampling signals with finite rate of innovation," *IEEE Trans. Signal Process.*, vol. 50, pp. 1417–1428, Jun. 2002.

[4] P. L. Dragotti, M. Vetterli, and T. Blu, "Exact sampling results for signals with finite rate of innovation using the Strang-Fix conditions and local kernels," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process. (ICASSP)*, Mar. 2005.

[5] P. L. Dragotti, M. Vetterli, and T. Blu, "Sampling moments and reconstructing signals of finite rate of innovation: Shannon meets strang-fix," *IEEE Trans. Signal Process.*, vol. 55, pp. 1741–1757, May 2007.

[6] G. Strang and G. Fix, "A Fourier analysis of the finite element variational method," *Construct. Aspects Funct. Anal.*, pp. 796–830, 1971.

[7] M. Unser and T. Blu, "Cardinal exponential splines: Part I—Theory and filtering algorithms," *IEEE Trans. Signal Process.*, vol. 53, pp. 1425–1438, Apr. 2005.

[8] J. Kusuma and V. K. Goyal, "Multichannel sampling of parametric signals with a successive approximation property," in *Proc. IEEE Int. Conf. Image Process. (ICIP)*, Oct. 2006, pp. 1265–1268.

[9] P. Stoica and R. Moses, *Introduction to Spectral Analysis*. Englewood Cliffs, NJ: Prentice-Hall, 2000.

[10] R. Blahut, *Theory and Practice of Error Control Codes*. Reading, MA: Addison-Wesley, 1983.

[11] J. M. N. Vieira and P. J. S. G. Ferreira, "Interpolation, spectrum analysis, error-control coding and fault-tolerant computing," in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Process. (ICASSP)*, Apr. 1997, vol. 3, pp. 1831–1834.

[12] B. Ettinger, N. Sarig, and Y. Yomdin, "Linear versus non-linear acquisition of step-functions," *J. Geo. Anal.*, vol. 18, no. 2, pp. 369–399, Apr. 2008.

[13] D. Batenkov, N. Sarig, and Y. Yomdin, "An algebraic reconstruction of piecewise-smooth functions from integral measurements," *Proc. Sampling Theory Applicat. (SAMPTA)*, May 2009.

[14] P. Milanfar, G. Verghese, W. Karl, and A. Willsky, "Reconstructing polygons from moments with connections to array processing," *IEEE Trans. Signal Process.*, vol. 43, pp. 432–443, Feb. 1995.

[15] M. Elad, P. Milanfar, and G. H. Golub, "Shape from moments—an estimation theory perspective," *IEEE Trans. Signal Process.*, vol. 52, pp. 1814–1829, Jul. 2004.

[16] P. Marziliano, M. Vetterli, and T. Blu, "Sampling and exact reconstruction of bandlimited signals with shot noise," *IEEE Trans. Inf. Theory*, vol. 52, pp. 2230–2233, May 2006.

[17] M. Vetterli, P. Marziliano, and T. Blu, "Sampling discrete-time piecewise bandlimited signals," in *Proc. Sampling Theory Applicat. Workshop*, 2001, pp. 97–102.

[18] D. Donoho, "Compressed sensing," *IEEE Trans. Inf. Theory*, vol. 52, pp. 1289–1306, Apr. 2006.

[19] E. J. Candes, J. Romberg, and T. Tao, "Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information," *IEEE Trans. Inf. Theory*, vol. 52, pp. 489–509, Feb. 2006.

[20] Y. C. Eldar, "Compressed sensing of analog signals in shift-invariant spaces," *IEEE Trans. Signal Process.*, vol. 57, pp. 2986–2997, Aug. 2009.

[21] M. Mishali and Y. C. Eldar, "Blind multi-band signal reconstruction: Compressed sensing for analog signals," *IEEE Trans. Signal Process.*, vol. 57, pp. 993–1009, Mar. 2009.

[22] Y. C. Eldar and T. Michaeli, "Beyond bandlimited sampling," *IEEE Signal Process. Mag.*, vol. 26, pp. 48–68, May 2009.

[23] M. Unser, A. Aldroubi, and M. Eden, "B-spline signal processing: Part I—Theory and Part II—Efficient design and applications," *IEEE Trans. Signal Process.*, vol. 41, pp. 821–848, Feb. 1993.

[24] J. D. Young, "Numerical applications of hyperbolic spline functions," *Logistics Rev.*, vol. 4, pp. 17–22, 1968.

[25] L. L. Schumaker, *Spline Functions: Basic Theory*. New York: Wiley, 1981.

[26] W. Dahmen and C. A. Micchelli, *On Theory and Application of Exponential Splines*, C. Chui, L. L. Schumaker, and F. Utreras, Eds. New York: Academic, 1987.

[27] B. J. McCartin, "Theory of exponential splines," *J. Approx. Theory*, vol. 66, pp. 1–23, 1991.

[28] I. Maravic and M. Vetterli, "Sampling and reconstruction of signals with finite rate of innovation in the presence of noise," *IEEE Trans. Signal Process.*, vol. 53, pp. 2788–2805, Aug. 2005.

[29] T. Blu, P. L. Dragotti, M. Vetterli, P. Marziliano, and L. Coulot, "Sparse sampling of signal innovations: Theory, algorithms and performance bounds," *IEEE Signal Process. Mag.*, vol. 25, pp. 31–40, Mar. 2008.

[30] V. F. Y. Tan and V. K. Goyal, "Estimating signals with finite rate of innovation from noisy samples: A stochastic algorithm," *IEEE Trans. Signal Process.*, vol. 56, pp. 5135–5146, Oct. 2008.

[31] J. Kusuma and V. K. Goyal, "On the accuracy and resolution of powersum-based sampling methods," *IEEE Trans. Signal Process.*, vol. 57, pp. 182–193, Jan. 2009.

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