Optimal Interpolation of a Fractional Brownian Motion Given its Noisy Samples

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Summary

We consider the problem of estimating a fractional Brownian motion known only from its noisy samples at the integers. We show that the optimal estimator can be expressed using a digital **Wiener-like filter** followed by a simple **time-variant correction** accounting for nonstationarity.

Moreover, we prove that this estimate lives in a **symmetric fractional spline** space and give a practical implementation for optimal upsampling of noisy fBm samples by integer factors.

What is a fractional Brownian motion?

A **non-stationary** zero-average **Gaussian** random process $W_\gamma(t)$ such that $W_\gamma(0)=0$ almost surely and whose increments are stationary with

$$\mathscr{E}\{|W_\gamma(t)-W_\gamma(t')|^2\} = C_\gamma|t-t'|^{2\gamma}$$

for some $\gamma \in]0,1[$ (Hurst exponent).

It can also be defined more explicitely, either by the **characteristic function** of any of its measurements $\langle W_\gamma, \psi\rangle$ (Gel'fand–Vilenkin's distributional approach)

$$\mathscr{E}\{e^{-j\langle W_\gamma,\psi\rangle}\} = \exp\left(-\frac{\varepsilon_\gamma^2}{4\pi}\int\frac{|\hat\psi(\omega)-\hat\psi(0)|^2}{|\omega|^{2\gamma+1}}\,\mathrm{d}\omega\right)$$

or through an **Itô stochastic integral** formulation

$$W_\gamma(t) = \frac{\varepsilon_\gamma}{\sqrt{2\pi}}\int\frac{e^{j\omega t}-1}{|\omega|^{\gamma+1/2}}\,\mathrm{d}W(\omega),$$

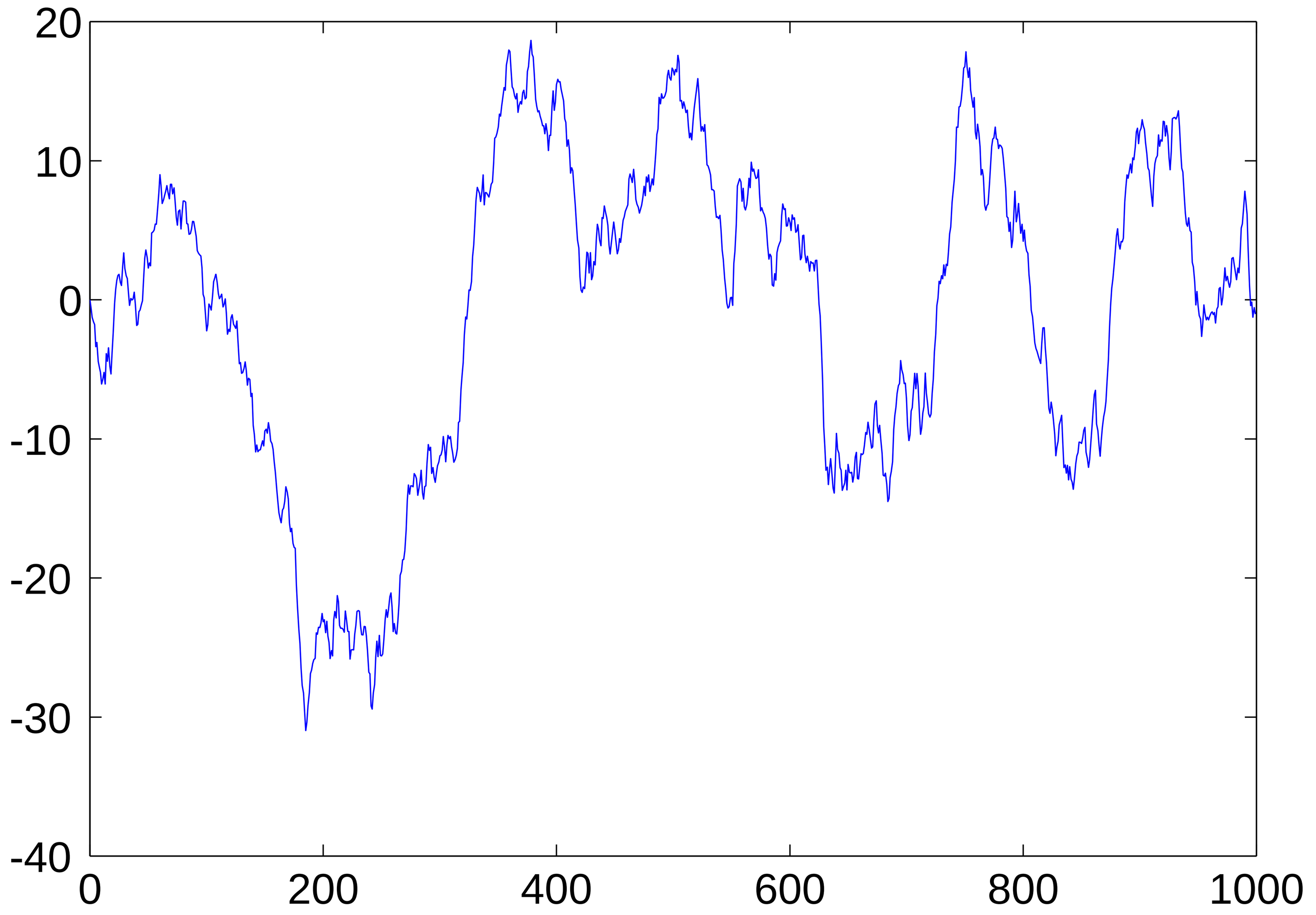
where $W(\omega)$ is a Wiener process, i.e., the usual Brownian motion, corresponding to $\gamma=1/2$. Here, $\varepsilon_\gamma = \sqrt{\Gamma(2\gamma+1)\sin(\pi\gamma)\,C_\gamma}$.

Filtering the integer samples of an fBm by

$$G(e^{j\omega}) = \left(\sum_{n\in\mathbb{Z}}\frac{1}{|\omega-2n\pi|^{2\gamma+1}}\right)^{-1}$$

gives a discrete white noise $\rightsquigarrow$ synthesis of an fBm by inverse filtering with $G(z)^{-1}$.

Example of fBm with $\gamma = 0.7$

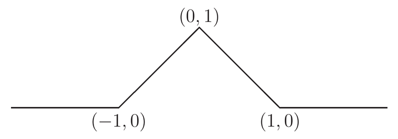


What is a fractional spline?

If $0<\alpha<2$, any function $f(t)$ that can be written as $f(t)=\sum_{n\in\mathbb{Z}}a_n|t-n|^\alpha$ is a fractional spline of degree α. A well-localized basis is the **symmetric fractional B-spline** $\beta_*^\alpha(t)$ characterized by the Fourier transform

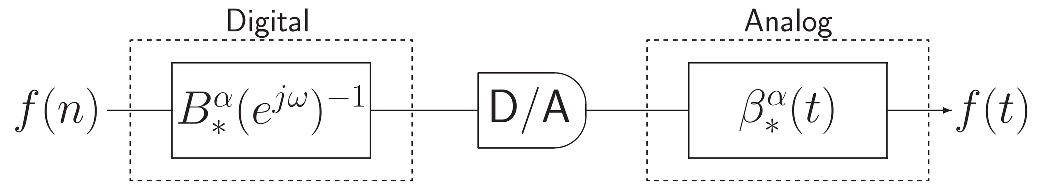
$$\hat\beta_*^\alpha(\omega) = \left|\frac{\sin(\omega/2)}{\omega/2}\right|^{\alpha+1}.$$

Example: if $\alpha=1$, $\beta_*^\alpha(t)$ is the triangle function



Many properties: valid M-scale **multi-resolution** analysis, regularity, short "equivalent" support, fast interpolation algorithm (Fourier).

Fractional spline interpolation



where the fractional spline interpolation prefilter is defined as

$$B_*^\alpha(e^{j\omega}) = \sum_{n\in\mathbb{Z}}\beta_*^\alpha(n)e^{-jn\omega} = \sum_{n\in\mathbb{Z}}\hat\beta_*^\alpha(\omega+2n\pi)$$

Fast algorithm for the computation of $B_*^\alpha(e^{j\omega})$.

What is the problem here?

Find the **optimal estimate** of an fBm $W_\gamma(t)$ from a series of noisy samples

$$y_k = W_\gamma(k) \quad + \underbrace{N(k)}_{\substack{\text{Gaussian stationary with autocorrelation } r_k\\ \text{independent from } W_\gamma(t)}}, \quad k\in\mathbb{Z}$$

Formal Bayesian solution: $W_{\gamma,\mathrm{est}}(t) = \mathscr{E}\{W_\gamma(t)\big|\{y_k\}_{k\in\mathbb{Z}}\}$. The main result of this paper is the explicit computation of this solution.

A fractional spline estimate

The optimal estimate of a **noisy fBm** with Hust exponent γ is a **fractional spline** of degree 2γ:

$$W_{\gamma,\mathrm{est}}(t) = \sum_{k\in\mathbb{Z}} c_k\beta_*^{2\gamma}(t-k)$$

with $c_k = \underbrace{h_k * y_k}_{\text{Wiener-like filtering}} \; - \; \lambda\, h_k * r_k.$

The constant λ is chosen in such a way that $W_{\gamma,\mathrm{est}}(0)=0$ and the Wiener-like filter is specified by

$$H(e^{j\omega}) = \frac{1}{B_*^{2\gamma}(e^{j\omega}) + \left|2\sin\frac{\omega}{2}\right|^{2\gamma+1}\frac{R(e^{j\omega})}{\varepsilon_\gamma^2}}$$

Originality: since the fBm is **not stationary**, the usual Wiener-Hopf denoising filter solution does no apply here. However, an equivalent Wiener filter arises from the solution, followed by a non-stationary correction which tends to zero for large t.

Example: resampling a noisy fBm

Given the noisy samples $y_k = W_\gamma(k)+N(k)$, find the optimal estimate of $W_\gamma(n/M)$ where $M\geq 2$ is an integer.

Solution: $W_{\gamma,\mathrm{est}}(n/M) = \sum_{k\in\mathbb{Z}} c_k\beta_*^{2\gamma}(n/M-k)$

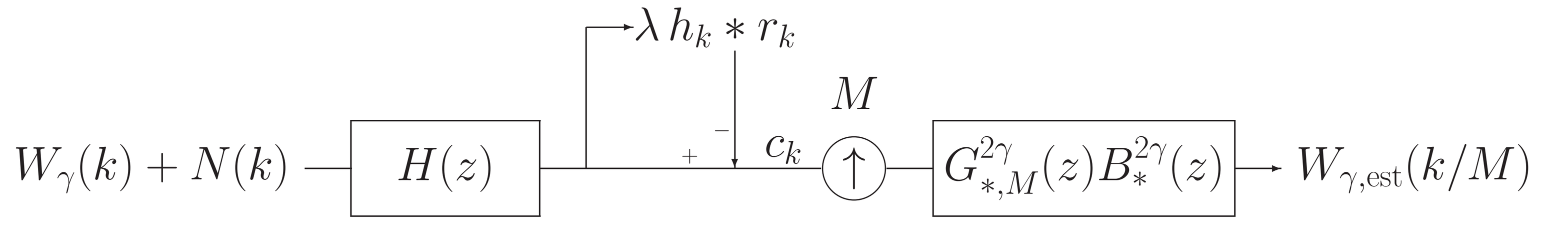
Implementation: using the scaling relation

$$\beta_*^{2\gamma}(n/M) = \sum_{k\in\mathbb{Z}} g_{*,M}^{2\gamma}[k]\beta_*^{2\gamma}(n-k) = (g_{*,M}^{2\gamma} * b_*^{2\gamma})[n]$$

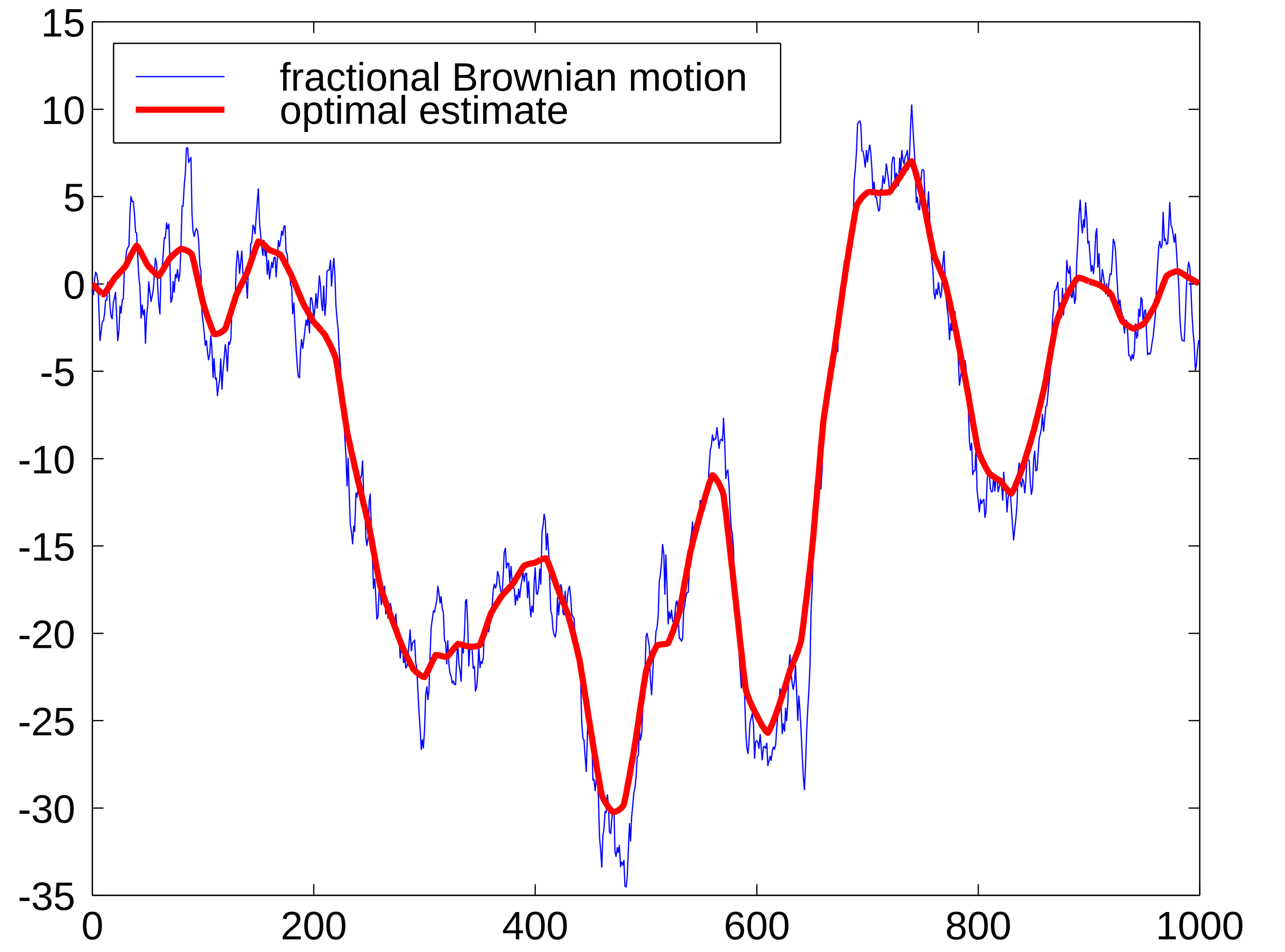
where $$G_{*,M}^{2\gamma}(e^{j\omega}) = M\left|\frac{\sin(M\omega/2)}{M\sin(\omega/2)}\right|^{2\gamma+1}$$

and the FFT algorithm.

Full algorithm



Denoising result with $\gamma = 0.6$



Conclusion

The result produced here is essentially theoretical, but brings a renewed insight into the estimation of nonstationary processes:

- The optimal estimation space is built using **shifts of the variogram** (a similar result with more constraints on the estimation is known in **Kriging** approaches); here, this space is a fractional spline space;
- The best approximation of an fBm is non-stationary as well, but can be decomposed into the sum of a stationary part (filter), and of a short-lived correction;
- The estimation space inherits the scale invariance of the fBm, a property that provides efficient **multiresolution** algorithms.