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Fractional Splines and Wavelets*

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Abstract. We extend Schoenberg's family of polynomial splines with uniform knots to all fractional degrees $\alpha > -1$. These splines, which involve linear combinations of the one-sided power functions $x_+^\alpha = \max(0, x)^\alpha$, are α-Hölder continuous for $\alpha > 0$. We construct the corresponding B-splines by taking fractional finite differences and provide an explicit characterization in both time and frequency domains. We show that these functions satisfy most of the properties of the traditional B-splines, including the convolution property, and a generalized fractional differentiation rule that involves finite differences only. We characterize the decay of the B-splines that are not compactly supported for nonintegral α's. Their most astonishing feature (in reference to the Strang–Fix theory) is that they have a fractional order of approximation $\alpha + 1$ while they reproduce the polynomials of degree $\lceil \alpha \rceil$. For $\alpha > -\frac{1}{2}$, they satisfy all the requirements for a multiresolution analysis of $\mathbf{L}^2$ (Riesz bounds, two-scale relation) and may therefore be used to build new families of wavelet bases with a continuously varying order parameter. Our construction also yields symmetrized fractional B-splines which provide the connection with Duchon's general theory of radial (m, s)-splines (including thin-plate splines). In particular, we show that the symmetric version of our splines can be obtained as the solution of a variational problem involving the norm of a fractional derivative.

Key words. splines, B-splines, wavelets, approximation theory, fractional derivatives, approximation order, multiresolution, Riesz basis, two-scale relation

AMS subject classifications. 41A15, 41A25, 65D07, 26A33

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1. Introduction.

A polynomial spline of order $L = n+1$ (or degree n) is a piecewise polynomial function of degree n that is constrained to be Hölder continuous of order n. Thus, its nth derivative, which is bounded, exhibits some isolated discontinuities at the *knots*, which are the joining points between the polynomial segments. This multiple differentiability constraint has one important implication, namely, that a spline has exactly one degree of freedom (or parameter) per knot. Polynomial splines with uniform knots were introduced by Schoenberg in his 1946 landmark paper, which sets the theoretical foundations for the subject [27, 30]. These functions now play a central role in approximation theory and numerical analysis. They have a number of desirable properties that make them useful in a variety of applications [14, 24, 4, 40].

Splines have also had a significant impact on the early development of the theory of the wavelet transform [36, 5, 19, 21, 23]. In this context, splines constitute a case apart for they yield the only wavelets that have an explicit analytical form. All other wavelet bases are defined indirectly through an infinite recursion (or an infinite product in Fourier domain) [13, 22, 35]. To date, four subfamilies of spline wavelets

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have been characterized explicitly: the orthogonal Battle–Lemarié wavelets [5, 19], the semiorthogonal spline wavelets [11, 39, 41], the biorthogonal splines [12], and the shift-orthogonal spline wavelets [42]. One notable property is that these splines—irrespective of their type—appear to have the best approximation properties among all known wavelet families: they yield the smallest asymptotic (scale-truncated) approximation error for a given order L [37, 38].

In this paper, we will extend the construction of polynomial splines to fractional degrees. Our new family will be indexed by a continuous parameter $\alpha > -1$, which represents the Hölder exponent of the fractional spline. This family interpolates the conventional splines which correspond to the special case where α is an integer. This kind of extension is similar to Duchon's generalization of the thin-plate splines [16], but the methods—as well as the context (cardinal splines versus radial basis functions)—are quite different. First, we consider a more constrained setting—univariate with equally spaced knots—which allows us to be much more explicit; the uniform grid in particular is required for constructing multiresolution wavelet bases. Second, our approach yields a larger class of splines than is possible with a purely variational formulation. In particular, Duchon's minimization technique cannot give the splines of even degree, whereas our method does, not to mention extensions to negative degrees α.

Our starting point is the construction of the fractional B-splines, which, to the best of our knowledge, have not been investigated before. The Fourier transform of the conventional B-splines of degree n is $\hat{\beta}^n_+(\omega) = \left((1-e^{-i\omega})/(i\omega)\right)^{n+1}$, and one natural approach to extend it to fractional orders is to use the same equation with α (noninteger) instead of n. Another possibility, which is more explicit, is to construct the B-splines from the (fractional) finite differences of one-sided power functions. We will see that both approaches are equivalent; in fact, we will show that the fractional splines share virtually all the properties of the conventional polynomial splines, except that the support of the B-splines for nonintegral α is no longer compact. In particular, they satisfy a two-scale relation and yield multiresolution analyses that are dense in $\mathbf{L}^2$ as soon as $\alpha > -\frac{1}{2}$. There is therefore no major difficulty in extending all standard wavelet constructions to the fractional case. However, there are also a few surprises in store concerning some standard notions in approximation and wavelet theory [34, 18, 35]. In this respect, the fractional splines' most notable idiosyncrasies are:

- Fractional splines, as their name should suggest, have a fractional order of approximation, a rather unusual property in approximation theory. Specifically, the approximation error at step size a, $\|f - \mathcal{P}_a f\|_{\mathbf{L}^2}$, decays like $a^{\alpha+1}$ as $a \to 0$. We will derive the asymptotic development of the $\mathbf{L}^2$ error and provide quantitative error bounds to substantiate this claim.
- For noninteger α, the fractional splines do not satisfy the Strang–Fix theory, which states the equivalence between the reproduction of polynomials of degree n and the order of approximation, which is one more than the degree ($L = n+1$) [34, 10, 18, 15]. We will see that fractional splines reproduce polynomials of degree n with $n-1 < \alpha \le n$ (or $n = \lceil \alpha \rceil$), while their order of approximation is $\alpha + 1$ (and not $\lceil \alpha \rceil + 1$, as one would expect).
- The fractional B-splines generate valid multiresolution analyses of $\mathbf{L}^2$ for $\alpha > -\frac{1}{2}$. However, for $-\frac{1}{2} < \alpha < 0$, their refinement filters $H(z)$ do not have the factor $1 + z$ which is usually required for the construction of valid wavelet bases [13, 22, 35]. Yet, the filters have the right vanishing property: $H(e^{i\pi}) = 0$, which guarantees the partition of unity condition [33] (except at the knots).

The paper is organized as follows. In section 2, we present an explicit construction of the fractional B-splines. The intent here is to offer some insight into what these splines really are. In section 3, we look at the fractional B-splines more closely and derive their most important mathematical properties (fractional differentiation rules, Riesz bounds, decay, and two-scale relation). In section 4, we characterize the ability of fractional splines to approximate functions and uncover some of their stranger properties. Finally, in section 5, we show that, when $\alpha > 0$, the symmetric fractional splines are the solutions of variational problems involving the minimization of the $\mathbf{L}^2$ norm of a fractional derivative. The appendices contain the more technical mathematical derivations; these are presented separately to improve the readability of the paper.

1.1. Notations and Definitions. One remarkable feature of fractional splines is that they involve an interesting mix of classical mathematics (Euler, Liouville) and more modern techniques derived from wavelet and approximation theory.

1.1.1. Gamma Function and Generalized Binomials. The gamma function, which was first studied by Euler, is defined as

$$\Gamma(u) = \int_0^{+\infty} x^{u-1} e^{-x} dx$$

for $u > 0$ and by the induction relation $\Gamma(u) = u^{-1}\Gamma(u+1)$ for $u < 0$. It has the property that $\Gamma(n+1) = n!$ and hence generalizes the factorial. The beta function is defined as

$$B(u, v) = \int_0^1 x^{u-1}(1-x)^{v-1} dx.$$

The relation between both integrals is given by Euler's formula

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}. \tag{1.1}$$

These formulae suggest the following generalization of the binomial coefficients:

$$\binom{u}{v} = \frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)}. \tag{1.2}$$

In particular, this definition implies that $\binom{u}{k} = 0$ for $k < 0$. Moreover, for $u \geq 0$, we have the well-known binomial theorem

$$(1+z)^u = \sum_{k \geq 0} \binom{u}{k} z^k. \tag{1.3}$$

This series converges for $|z| \leq 1$, since

$$\binom{u}{k} \approx (-1)^{k+1} \frac{\Gamma(u+1) \sin \pi u}{\pi k^{u+1}}$$

as $k \to \infty$ by Stirling's formula. When $u = n$ (integer), $\binom{n}{k} = 0$ for $k \geq n+1$ and one recovers the standard binomial expansion.

In addition to these gamma-related functions, we will also need Riemann's zeta function, defined by

$$\zeta(\alpha) = \sum_{n \geq 1} \frac{1}{n^\alpha} \tag{1.4}$$

for $\alpha > 1$.

1.1.2. Fractional Derivatives. Liouville's generalization of differentiation for fractional orders is $D^\alpha f = g_{-\alpha} * f$ (cf. [20]), where

$$g_\alpha(x) = \frac{x_+^{\alpha-1}}{\Gamma(\alpha)}$$

and where

$$x_+^\alpha = \begin{cases} x^\alpha, & x \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

is the one-sided power function; the convolution has to be taken in the sense of distributions. Note that unlike integer differentiation, not every distribution has a fractional derivative; for instance, $f(x) = e^{-x}$.

In order to correctly interpret these derivatives in the Fourier domain, we first define the fractional power of a complex variable z as $z^\alpha = |z|^\alpha e^{i\alpha \arg(z)}$ with $i = \sqrt{-1}$ and $\arg(z) \in [-\pi, \pi[$. The usual composition property of the power function takes a more restricted form in the complex plane: specifically, one has $(z_1 z_2)^\alpha = z_1^\alpha z_2^\alpha$ *only if* $\arg(z_1) + \arg(z_2) \in [-\pi, \pi[$ when $z_1 z_2 \neq 0$, while $z^{\alpha_1} z^{\alpha_2} = z^{\alpha_1 + \alpha_2}$ remains always true.

The Fourier domain equivalent of Liouville's definition of the fractional derivative is $\int D^\alpha f(x) e^{-i\omega x} dx = (i\omega)^\alpha \hat{f}(\omega)$, where $\hat{f}(\omega)$ denotes the Fourier transform of f and where $(i\omega)^\alpha$ has to be evaluated in accordance with our convention. This is the expected generalization of the well-known formula for integer exponents. However, due to the discontinuity of $(i\omega)^\alpha$ near 0 for nonintegral orders, the fractional derivative is in general a nonlocal operation that tends to produce slowly decaying functions.

1.1.3. Some Useful Fourier Transforms. Classical Fourier theory can be extended to the *tempered distributions* as defined by Schwartz [31]. The Fourier transform pairs $(u(x) \longleftrightarrow \hat{u}(\omega))$ that are useful for our purpose are:

$$\begin{aligned} x_+^\alpha &\longleftrightarrow \frac{\Gamma(\alpha+1)}{(i\omega)^{\alpha+1}} \text{ if } \alpha \text{ is not an integer;} \\ x_+^n &\longleftrightarrow \frac{\Gamma(n+1)}{(i\omega)^{n+1}} + i^n \pi \delta^{(n)}(\omega) \text{ if } n \text{ is a positive integer;} \\ |x|^\alpha &\longleftrightarrow -\frac{2\sin(\frac{\pi}{2}\alpha)\Gamma(\alpha+1)}{|\omega|^{\alpha+1}} \text{ if } \alpha > -1 \text{ is not an even integer;} \\ x^{2n} \log|x| &\longleftrightarrow \frac{(-1)^{n+1}\pi\Gamma(2n+1)}{|\omega|^{2n+1}} \text{ if } n \text{ is a positive integer.} \end{aligned} \tag{1.5}$$

The last pair requires the following definition of the distribution $|x|^{-2n-1}$:

$$\frac{1}{|x|^{2n+1}} = D^{2n+1}\left(\frac{\operatorname{sgn}(x)(\log|x| + \gamma)}{\Gamma(2n+1)}\right),$$

where γ is Euler's constant (i.e., $\gamma = \lim_{n\to\infty} \sum_{k=1}^n \frac{1}{n} - \log n$) [9].

2. Construction of Fractional Splines. In this section, we propose a formal construction that proceeds by analogy with the polynomial spline case. Our principal goal here is to motivate our definition of the fractional B-splines and to establish some of their elementary properties. The more rigorous mathematical analysis will be given in section 3.

2.1. Preliminaries. The natural building blocks for the fractional splines are Liouville's one-sided power functions x_+^α, which have precisely one singularity of order α (Hölder exponent) at the origin. Thus, we may think of fractional splines of degree α with the increasing sequence of knots $\{x_k\}_{k\in\mathbb{Z}}$ as functions that can be written in the following form:

$$s^\alpha(x) = \sum_{k\in\mathbb{Z}} a_k\,(x-x_k)_+^\alpha\,,$$

where the x_k's are the knots of the spline. This representation has some obvious problems associated with it when we extend it to the whole real axis because the one-sided power functions are unbounded. However, it offers insight into what the fractional splines really are.

From now on, we will exclusively consider fractional splines with knots at the integers. The relevant tool in this context is the fractional forward finite difference operator, which we define as

$$\Delta_+^\alpha f(x) = \sum_{k\geq 0}(-1)^k\binom{\alpha}{k} f(x-k). \tag{2.1}$$

This is a *convolution* operator, which has a more straightforward interpretation in the Fourier domain:

$$\hat{\Delta}_+^\alpha(\omega) = (1-e^{-i\omega})^\alpha = \sum_{k\geq 0}(-1)^k\binom{\alpha}{k} e^{-i\omega k}.$$

The expansion on the right-hand side is a direct application of the generalized binomial formula (1.3), which ensures that $\Delta_+^{\alpha_1}\Delta_+^{\alpha_2} = \Delta_+^{\alpha_1+\alpha_2}$. It also guarantees that our operator coincides with the conventional one when α is an integer.

2.2. Fractional B-Splines. By analogy with the classical B-splines, we define the fractional causal B-splines by taking the $(\alpha+1)$th fractional difference of the one-sided power function

$$\beta_+^\alpha(x) \stackrel{\text{def}}{=} \frac{1}{\Gamma(\alpha+1)}\,\Delta_+^{\alpha+1} x_+^\alpha = \frac{1}{\Gamma(\alpha+1)}\sum_{k\geq 0}(-1)^k\binom{\alpha+1}{k}(x-k)_+^\alpha\,. \tag{2.2}$$

In section 3.2, we will show that these functions are in $\mathbf{L}^1$ for $\alpha > -1$ and in $\mathbf{L}^2$ for $\alpha > -\frac{1}{2}$; we will also prove that they decay proportionally to $|x|^{-\alpha-2}$ (see Theorem 3.1). Some examples of fractional B-splines are shown in Figure 2.1. While they seem to be decaying reasonably rapidly, they are not compactly supported unless α is an integer, in which case we recover the classical B-splines. In general, they do not have an axis of symmetry either.

PROPOSITION 2.1. *The fractional causal B-splines satisfy the convolution property*

$$\beta_+^{\alpha_1} * \beta_+^{\alpha_2} = \beta_+^{\alpha_1+\alpha_2+1}.$$

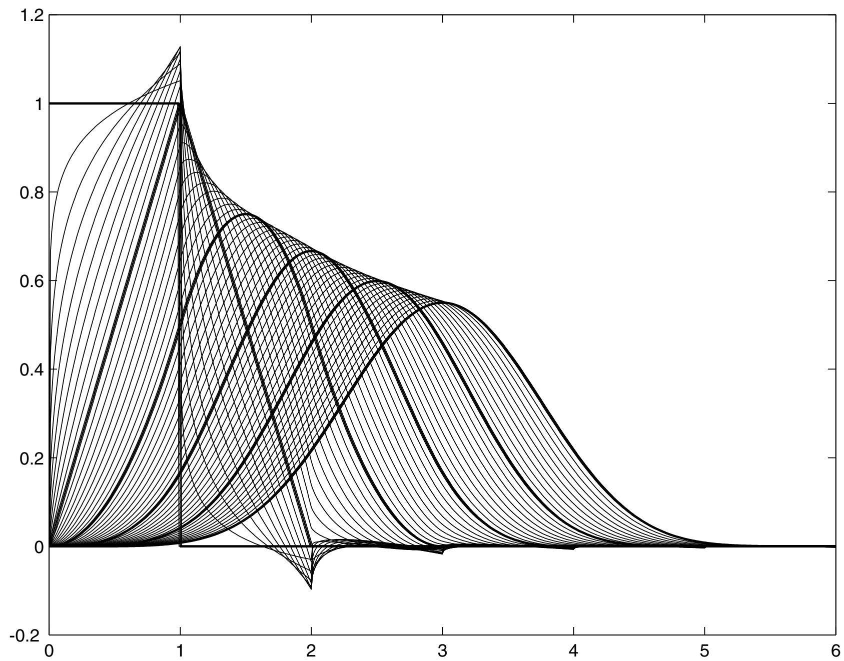

Fig. 2.1 *The fractional B-splines with $\alpha \geq 0$. These functions interpolate the conventional B-splines which are represented using a thicker line.*

Proof. Let us consider the convolution integral

$$x_+^{\alpha_1-1} * x_+^{\alpha_2-1} = \int_0^x y^{\alpha_1-1}(x-y)^{\alpha_2-1}dy,$$

which is obviously 0 for $x \leq 0$. For $x > 0$, we make the change of variable $u = y/x$, and rewrite the integral in terms of the beta function. This provides $x_+^{\alpha_1-1} * x_+^{\alpha_2-1} = B(\alpha_1, \alpha_2)x^{\alpha_1+\alpha_2-1}$ for $x > 0$, and 0 for $x \leq 0$. We then use Euler's formula (1.1) to show that the one-sided power functions satisfy the convolution property

$$\frac{x_+^{\alpha_1-1}}{\Gamma(\alpha_1)} * \frac{x_+^{\alpha_2-1}}{\Gamma(\alpha_2)} = \frac{x_+^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)}.$$

The result in Proposition 2.1 then follows almost immediately from the definition (2.2) of the fractional B-splines, thanks to the commutativity of the convolution operator, and the composition rule of the Δ operator, namely, $\Delta_+^{\alpha_1}\Delta_+^{\alpha_2} = \Delta_+^{\alpha_1+\alpha_2}$. □

For the sake of completeness, we also introduce the reversed versions of these functions: the anticausal B-splines of degree α,

$$\beta_-^\alpha(x) \stackrel{\text{def}}{=} \Delta_-^{\alpha+1}\left(\frac{x_-^\alpha}{\Gamma(\alpha+1)}\right) = \beta_+^\alpha(-x),$$

where $x_-^\alpha = (-x)_+^\alpha$ and where Δ_-^α denotes the fractional *backward* difference operator

$$\Delta_-^\alpha f(x) = \sum_{k\geq 0}(-1)^k \binom{\alpha}{k} f(x+k).$$

Our symbolism is such that all formulae for the causal fractional B-splines carry over directly to the noncausal ones by simply replacing "+" by "−."

PROPOSITION 2.2. *The fractional B-splines satisfy the induction equation*

$$\beta_+^\alpha(x) = \frac{x}{\alpha}\beta_+^{\alpha-1}(x) + \frac{\alpha+1-x}{\alpha}\beta_+^{\alpha-1}(x-1) \tag{2.3}$$

for all $\alpha > 0$.

For α integer, (2.3) is the well-known recurrence relation for B-splines on a uniform grid [14].

Proof. The relation derives from the identity

$$\Delta_+^{\alpha+1}\{xf(x)\} = (\alpha+1)\Delta_+^\alpha f(x) + (x-\alpha-1)\Delta_+^{\alpha+1} f(x),$$

which is easily established using the definition (2.1) of Δ_+^α and the induction relation $k\binom{\alpha}{k} = \alpha\binom{\alpha-1}{k-1}$. Letting $f(x) = \frac{x_+^{\alpha-1}}{\Gamma(\alpha+1)}$ we then obtain

$$\beta_+^\alpha(x) = \frac{\alpha+1}{\alpha}\beta_+^{\alpha-1}(x) + \frac{x-\alpha-1}{\alpha}\Delta_+\beta_+^{\alpha-1}(x),$$

from which (2.3) follows immediately. □

2.3. Symmetric Fractional B-Splines. To symmetrize the construction and to be in the position to calculate fractional B-spline inner products, we define the symmetric B-splines of fractional degree α:

$$\beta_*^\alpha = \beta_+^{\frac{\alpha-1}{2}} * \beta_-^{\frac{\alpha-1}{2}}. \tag{2.4}$$

Since β_+^α and β_-^α are in $\mathbf{L}^1$ (see Theorem 3.2 below), this convolution has a meaning for $\alpha > -1$ (the convolution of two $\mathbf{L}^1$ functions is in $\mathbf{L}^1$ as well).

These symmetric fractional splines may also be specified in the Fourier domain, where they have the convenient form (cf. section 3.1 below)

$$\hat{\beta}_*^\alpha(w) = \left|\frac{\sin(\omega/2)}{\omega/2}\right|^{\alpha+1}. \tag{2.5}$$

To facilitate the calculation of the corresponding inverse Fourier transform, we were tempted to introduce the corresponding analogs $\left|\begin{smallmatrix} r \\ k \end{smallmatrix}\right|$ of the generalized binomial coefficients; these are defined through the following generating function:

$$|1+z|^\alpha = \sum_{k\in\mathbb{Z}} \begin{vmatrix} \alpha \\ k \end{vmatrix} z^k \quad \text{for } z = e^{-i\omega},$$

which is convergent only on the unit circle. The following result, which is derived in Appendix A, gives the explicit form of these quantities.

LEMMA 2.3. *The modified binomial coefficients are symmetrical and satisfy*

$$\begin{vmatrix} \alpha \\ k \end{vmatrix} = \binom{\alpha}{k+\frac{\alpha}{2}}, \tag{2.6}$$

where the right-hand-side term is defined through (1.2).

Thus, the modified binomial coefficients are recentered versions of the generalized ones. They can only vanish if α is even, in which case the sequence is finite.

By analogy with our definition of nonsymmetric fractional B-splines, we define

$$x_*^\alpha = \begin{cases} |x|^\alpha & \text{if } \alpha > -1 \text{ is not an even integer,} \\ x^\alpha \log|x| & \text{if } \alpha \in 2\mathbb{N}. \end{cases} \tag{2.7}$$

We also introduce the symmetric finite difference operator Δ_*^α, whose frequency response is $|1 - e^{-i\omega}|^\alpha$.

Using the Fourier correspondences (1.5) in (2.5), we thus find an explicit time domain formula for the symmetric fractional splines.

THEOREM 2.4. *The centered fractional B-splines of degree* α *are given by*

(i) $\alpha \geq -1$ *and* $\alpha \neq 2n$ *(not even):*

$$\begin{aligned} \beta_*^\alpha(x) &= \frac{-1}{2\sin(\frac{\pi}{2}\alpha)\Gamma(\alpha+1)}\,\Delta_*^{\alpha+1} x_*^\alpha \\ &= \frac{1}{2\sin(\frac{\pi}{2}\alpha)\Gamma(\alpha+1)} \sum_{k\in\mathbb{Z}} (-1)^{k+1} \left|\begin{matrix}\alpha+1 \\ k\end{matrix}\right| |x-k|^\alpha; \end{aligned} \tag{2.8}$$

(ii) $\alpha = 2n$ *(even)*

$$\begin{aligned} \beta_*^{2n}(x) &= \frac{(-1)^{n+1}}{\pi\Gamma(2n+1)}\,\Delta_*^{2n+1} x_*^{2n} \\ &= \frac{(-1)^n}{2n!\,\pi} \sum_{k\in\mathbb{Z}} (-1)^{k+1} \left|\begin{matrix}2n+1 \\ k\end{matrix}\right| |x-k|^{2n} \log|x-k|. \end{aligned} \tag{2.9}$$

Some examples of the fractional centered B-splines are shown in Figure 2.2. Similar to their causal counterparts, they are α-Hölder continuous with knots at the integers; they are not compactly supported either unless n is odd. The most notable difference is that our centered B-splines are constructed using the integer shifts of $|x|^\alpha$ (and of $x^\alpha \log|x|$ when α is an even integer) rather than x_+^α. Also note that they coincide with the standard centered B-splines only when the degree is *odd*; this is because of the absolute value in (2.5), which creates a discontinuity in the Fourier domain when α is even. For comparison, Schoenberg's centered B-splines of even degree have knots at the half integers; they therefore span different spaces.

Since $|x|^\alpha = x_-^\alpha + x_+^\alpha$, the symmetrized B-splines belong to the multiwavelet space spanned by the integer shifts of the two (nonsymmetric) functions β_+^α and β_-^α, unless $\alpha \in 2\mathbb{N}$. Thus, we may think of the space generated by the β_*^α as a "symmetrization" of the spline spaces generated by β_+^α and β_-^α.

Interestingly, the elementary functions that generate the symmetric fractional B-splines ($|x|^\alpha$ and $|x|^{2n}\log|x|$) are the same as those that appear in Duchon's generalized theory of thin-plate splines [16]. This is no coincidence; the link will be made explicit in section 5, where the symmetric fractional splines are shown to minimize some Duchon seminorm. Hence, we may think of the centered fractional B-splines as a (univariate) way of localizing these radial basis functions on a uniform grid.

We also note that Rabut [25] briefly suggests the possibility of generalizing his polyharmonic B-splines for noninteger orders using a multidimensional Fourier domain formula which is compatible with (2.5); however, he did not pursue this idea much further.

One potential problem with the expressions given in Theorem 2.4 is that the series are slowly convergent. Indeed, by using Stirling's formula, one shows that

$$\left|\begin{matrix}\alpha \\ k\end{matrix}\right| \approx \frac{\sin(\frac{\pi}{2}\alpha)\Gamma(\alpha+1)}{\pi} \times \frac{(-1)^{k+1}}{k^{\alpha+1}},$$

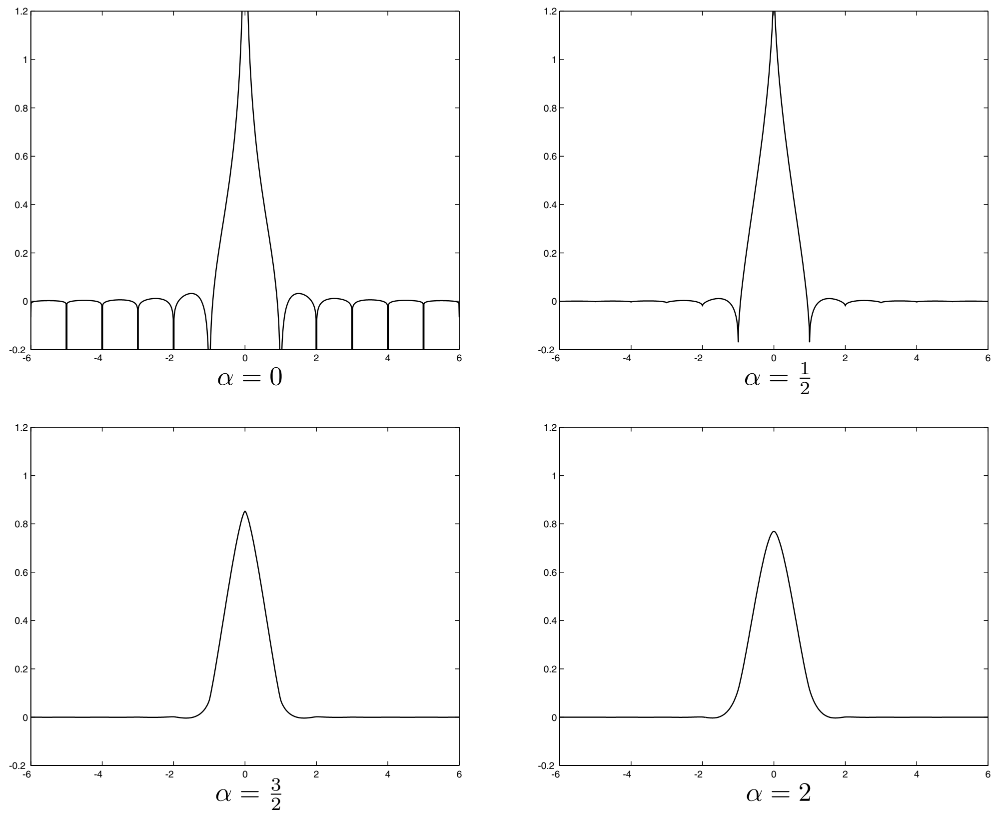

Fig. 2.2 *Examples of symmetric fractional B-splines of increasing regularity. For $\alpha \le 0$, the B-spline is infinite at the integers, whereas for higher α it is continuous everywhere.*

which implies that (i) converges as $\sum_{k\ge1} k^{-2}$ and (ii) as $\sum_{k\ge1} k^{-2}\log k$, independently of the order of the spline.

Fortunately, it is possible to improve the convergence of these expressions by using standard acceleration techniques, as shown in the following theorem.

THEOREM 2.5. *Let $\beta^\alpha_{*,N}(x)$ and $\beta^{2n}_{*,N}(x)$ correspond to the truncated sum ($|k| \le N$) approximations of* (2.8) *and* (2.9), *respectively. Then, one has the following asymptotic relations:*

(i) $\alpha > -1$ *and* $\alpha \ne 2n$,

$$\beta^\alpha_*(x) = \beta^\alpha_{*,N}(x) + \underbrace{\frac{\alpha+1}{\pi\tan\frac{\pi}{2}\alpha}\left(\frac{1}{N} - \frac{1}{2N^2}\right)}_{C_N^\alpha} + O\left(\frac{1}{N^3}\right); \tag{2.10}$$

(ii) $\alpha = 2n$,

$$\beta^{2n}_*(x) = \beta^{2n}_{*,N}(x) + \underbrace{\frac{4n+2}{\pi^2}\left(\frac{1+\log N}{N} - \frac{\log N}{2N^2}\right)}_{C_N^{2n}} + O\left(\frac{\log N}{N^3}\right). \tag{2.11}$$

Thus, the correction term C_N^α given above can be added to the partial sums to achieve much better convergence rates, namely, N^{-3} and $N^{-3}\log N$. This result is straightforwardly (but tediously) proved by verifying that $\beta^\alpha_{*,N+1}(x) + C^\alpha_{N+1} - (\beta^\alpha_{*,N}(x)+C^\alpha_N)$ is $O(N^{-4})$ if $\alpha \neq 2n$ and $O(N^{-4}\log N)$ if $\alpha = 2n$. Interestingly, these correction terms do not involve the value of x, which means that they can be seen as uniform biases. Using these baseline corrections turned out to be most useful for producing the graphs in Figure 2.2 in a computationally efficient way.

The symmetric splines satisfy a recurrence relation similar to (2.3), except that the induction jumps by steps of two instead of one.

PROPOSITION 2.6. *The symmetric fractional B-splines satisfy the induction equation*

$$\begin{aligned}\beta_*^\alpha(x) &= \frac{\left(x+\frac{\alpha+1}{2}\right)^2}{\alpha(\alpha-1)}\beta_*^{\alpha-2}(x+1) + \frac{\left(x-\frac{\alpha+1}{2}\right)^2}{\alpha(\alpha-1)}\beta_*^{\alpha-2}(x-1) \\ &\quad -2\frac{x^2+\frac{1-\alpha^2}{4}}{\alpha(\alpha-1)}\beta_*^{\alpha-2}(x)\end{aligned} \tag{2.12}$$

for all $\alpha > 1$.

Proof. Using the induction relations for the modified binomials $\left|\begin{smallmatrix}\alpha\\k\end{smallmatrix}\right|$,

$$\frac{2k}{\alpha}\begin{vmatrix}\alpha\\k\end{vmatrix} = \begin{vmatrix}\alpha-2\\k-1\end{vmatrix} - \begin{vmatrix}\alpha-2\\k+1\end{vmatrix} \quad \text{and} \quad \left(k^2-\frac{\alpha^2}{4}\right)\begin{vmatrix}\alpha\\k\end{vmatrix} = -\alpha(\alpha-1)\begin{vmatrix}\alpha-2\\k\end{vmatrix},$$

we readily verify that

$$\begin{aligned}\Delta_*^\alpha\{x^2 f(x)\} = &\left(x^2+\frac{\alpha^2}{4}\right)\Delta_*^\alpha f(x) \\ &-\alpha(\alpha-1)\Delta_*^{\alpha-2}f(x) - \alpha x\Delta_*^{\alpha-2}\{f(x+1)-f(x-1)\}\end{aligned}$$

by applying the direct definition of Δ_*^α. From this, we get the relation between $\beta_*^{\alpha-1}$ and $\beta_*^{\alpha-3}$ by letting $f(x) = x_*^{\alpha-3}$, which is equivalent to (2.12) provided that we substitute α by $\alpha+1$. ☐

3. Characterization of Fractional B-Splines.

Most of our characterization of the properties of the fractional B-spline will be carried out in the Fourier domain. In particular, we will show that they form a Riesz basis and investigate their decay and multiresolution properties. We use the generic notation $\beta^\alpha(x)$ to specify any one of the fractional B-splines ($\beta_+^\alpha(x)$, $\beta_-^\alpha(x)$, or $\beta_*^\alpha(x)$).

3.1. Fourier Transform.

The Fourier transform of $\beta_+^\alpha(x)$ is determined through a standard calculation that involves distributions because of the one-sided power functions. The final result is

$$\hat\beta_+^\alpha(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^{\alpha+1}, \tag{3.1}$$

which holds in the distributional sense. This equation is obviously compatible with the convolution property of Proposition 2.1. For $\alpha > 0$, the inverse Fourier transform can be computed in the usual sense, so that we have for every value of $x \in \mathbb{R}$

$$\beta_+^\alpha(x) = \frac{1}{2\pi}\int\left(\frac{1-e^{-i\omega}}{i\omega}\right)^{\alpha+1} e^{i\omega x}d\omega.$$

Using (3.1), one easily checks that $\hat{\beta}_{-}^{\alpha}(\omega) = \overline{\hat{\beta}_{+}^{\alpha}(\omega)}$ and thus that $\hat{\beta}_{*}^{\alpha}(\omega) = |\hat{\beta}_{+}^{\alpha}(\omega)|$, as claimed in (2.5).

As an interesting application of (3.1), we can establish the following fractional differentiation rule for B-splines:

$$D^{\gamma}\beta_{+}^{\alpha} = \Delta_{+}^{\gamma}\beta_{+}^{\alpha-\gamma}. \tag{3.2}$$

Note that one of the primary reasons for the success of polynomial splines in applications is precisely that they can be differentiated very simply by taking finite differences [24]. Here, we see that this property generalizes nicely to the fractional case. A similar property holds true for β_{-}^{α} and β_{*}^{α} if we substitute the index "+" in (3.2) by, respectively, "−" or "∗," and if we replace the differentiation operator by D_{-}^{α} or D_{*}^{α}, where $D_{-}^{\alpha}f \leftrightarrow (-i\omega)^{\alpha}\hat{f}$ and $D_{*}^{\alpha}f \leftrightarrow |\omega|^{\alpha}\hat{f}$. Note that, unlike D^{α} (which we could rename D_{+}^{α} for the sake of consistency), these operators do not, in general, coincide with the usual differentiation, except when α is even.

Since $|\hat{\beta}_{+}^{\alpha}(\omega)| = \hat{\beta}_{*}^{\alpha}(\omega)$ is bounded and decays like $|\omega|^{-\alpha}$ when $\omega \to \infty$, we can already claim that the fractional B-splines are in $\mathbf{L}^2$ for $\alpha > -\frac{1}{2}$: this result will be discussed again in Theorem 3.2 and proved directly using the decay rate of the β^{α}. More generally, one has

$$\beta^{\alpha} \in \mathbf{W}_2^r \quad \text{for all } r < \alpha + \frac{1}{2} \tag{3.3}$$

so that the critical Sobolev exponent of the fractional splines is $r_{\max} = \alpha + \frac{1}{2}$, that is, one-half more than their Hölder exponent α.

3.2. Decay. The only shortcoming of the fractional B-splines is their lack of compact support. It is therefore crucial to characterize their decay.

THEOREM 3.1. *For all $\alpha > -1$, there exist positive constants K, C_{α} such that*

$$|\beta^{\alpha}(x)| \le \frac{K_{\alpha}\{[x]\}_{*}^{\alpha} + C_{\alpha}}{1+|x|^{\alpha+2}}, \tag{3.4}$$

where $[x]$ is defined by $[x] = \inf_{n\in\mathbb{Z}} |x-n| = \left|x - \lfloor x+\frac{1}{2}\rfloor\right|$. More precisely, when $\alpha > 0$, we have

$$\beta_{+}^{\alpha}(x) = \frac{\Gamma(\alpha+2)\sin\pi\alpha}{\pi x^{\alpha+2}} \sum_{n\ge 1} \frac{e^{2ni\pi x}}{(2ni\pi)^{\alpha+1}} + o\left(\frac{1}{x^{\alpha+2}}\right), \tag{3.5}$$

$$\beta_{*}^{\alpha}(x) = -\frac{2\Gamma(\alpha+2)\cos(\frac{\pi}{2}\alpha)}{\pi x^{\alpha+2}} \sum_{n\ge 1} \frac{\cos(2n\pi x)}{(2n\pi)^{\alpha+1}} + o\left(\frac{1}{x^{\alpha+2}}\right) \tag{3.6}$$

when x tends to $+\infty$.

The proof is rather technical and is given in Appendix B. Note that Buhmann gives a similar result in [8, Theorem 6] for the decay of the *interpolating* basis functions within the framework of n-dimensional radial basis functions, which is more general but excludes our nonsymmetric splines. In this respect, we observe that β_{*}^{α} for $\alpha \le 0$ does not satisfy the hypotheses made in [8].

It follows from the characterization of their decay that the fractional B-splines belong to the classical integration spaces $\mathbf{L}^1$ and $\mathbf{L}^2$.

THEOREM 3.2. *The fractional splines β_{+}^{α} are in $\mathbf{L}^1$ for all $\alpha > -1$. Moreover, for $\alpha > -\frac{1}{2}$ they are in $\mathbf{L}^2$ as well.*

Proof. This is a direct consequence of (3.4), which shows that β_{+}^{α} is bounded by an $\mathbf{L}^1$ function when $\alpha > 0$, and by an $\mathbf{L}^2$ function when $\alpha > -\frac{1}{2}$. □

3.3. Fractional Spline Spaces and Riesz Bounds. We are now in the position to specify the fractional splines in a stable and rigorous fashion using the B-splines as basis functions. The basic space of fractional splines of degree α with knots at the integers is defined as

$$\mathbf{S}_+^\alpha = \left\{ s : \exists c \in \ell^2, \qquad s(x) = \sum_{k\in\mathbb{Z}} c(k)\beta_+^\alpha(x-k) \right\}. \tag{3.7}$$

Similarly, we may consider the spline subspaces $\mathbf{S}_-^\alpha$, $\mathbf{S}_*^\alpha$ generated by β_-^α, β_*^α, respectively, or, more generally, the space $\mathbf{S}^\alpha$ when the spline type is implicit.

PROPOSITION 3.3. *For $\alpha > -\frac{1}{2}$, the fractional B-spline of degree α generates a Riesz basis of $\mathbf{S}^\alpha$. Specifically, one has the following ℓ^2-$\mathbf{L}^2$ norm equivalence:*

$$\forall c \in \ell^2, \qquad A_\alpha \|c\|_{\ell^2} \le \|\sum_{k\in\mathbb{Z}} c(k)\beta^\alpha(x-k)\|_{\mathbf{L}^2} \le B_\alpha \|c\|_{\ell^2}, \tag{3.8}$$

where A_α and B_α are two constants such that

$$A_\alpha \ge \left(\frac{2}{\pi}\right)^{\alpha+1}, \qquad B_\alpha \le \left(1 + \frac{2\zeta(2\alpha+2)}{\pi^{2\alpha+2}}\left(1 - \frac{1}{2^{2\alpha+2}}\right)\right)^{\frac{1}{2}}.$$

Proof. Finding the Riesz bounds for β^α is equivalent to bounding

$$a(\omega) = \sum_n |\hat{\beta}^\alpha(\omega + 2n\pi)|^2$$

from above and below [3]. This function is 2π-periodic and symmetric, so we can restrict its study to $\omega \in [0,\pi]$. In particular, we have $a(\omega) \ge |\operatorname{sinc}\frac{\omega}{2}|^{2\alpha+2} \ge (\frac{2}{\pi})^{2\alpha+2}$, since $\operatorname{sinc}\frac{\omega}{2}$ is strictly decreasing over $[0,\pi]$; this provides A_α.

Since $\sup_{|\omega|\le\pi} |\beta^\alpha(\omega + n\pi)|^{2\alpha+2} \le \big(\pi(2|n|-1)\big)^{-2\alpha-2}$ for $n \neq 0$, we also get

$$a(\omega) \le 1 + \frac{1}{\pi^{2\alpha+2}} \sum_{n\in\mathbb{Z}} \frac{1}{|2n-1|^{2\alpha+2}} = 1 + \frac{2\zeta(2\alpha+2)}{\pi^{2\alpha+2}}\left(1 - \frac{1}{2^{2\alpha+2}}\right),$$

which gives the bound for B_α. □

This result ensures that the B-spline representation (3.7) is stable and that the fractional spline spaces are well-defined (closed) subspaces of $\mathbf{L}^2$. Starting from the B-splines, it is then easy, using the method described in [3], to generate other equivalent bases of these spaces with specific properties, for instance, orthogonality or interpolation. While an orthogonal basis always exists, the same is not necessarily true for the interpolating one (fundamental spline). For instance, it is well known that the polynomial spline interpolator is ill defined when the degree n is even and the knots are on the integers. Interestingly, the lower bound on A_α guarantees the existence of the fractional spline interpolators in the spaces $\mathbf{S}_*^\alpha$ (symmetric splines) for any $\alpha > -\frac{1}{2}$, including even ones. Also note that the orthogonal and interpolating splines all converge to $\frac{\sin(\pi x)}{\pi x}$ (the ideal lowpass filter) as the fractional degree α tends to infinity. This comes as a direct consequence of the general convergence theorems in [3].

3.4. Two-Scale Relation. The fractional B-splines have all the required multiresolution properties for the construction of wavelet bases. In particular, they satisfy the two-scale relation

$$\beta^\alpha\left(\frac{x}{2}\right) = \sum_{k\in\mathbb{Z}} h^\alpha(k)\beta^\alpha(x-k).$$

This equation can be established by direct manipulation of the time domain formulae. However, the simplest approach is to take the ratio $2\hat{\beta}^\alpha(2\omega)/\hat{\beta}^\alpha(\omega)$ to get the transfer function of the refinement filter $\hat{h}^\alpha(\omega)$, which turns out to be 2π-periodic. Specifically, we find that

$$h_+^\alpha(k) = \frac{1}{2^\alpha}\binom{\alpha+1}{k} \quad \longleftrightarrow \quad \hat{h}_+^\alpha(\omega) = 2\left(\frac{1+e^{-i\omega}}{2}\right)^{\alpha+1} \tag{3.9}$$

and

$$h_*^\alpha(k) = \frac{1}{2^\alpha}\left|\begin{matrix}\alpha+1\\ k\end{matrix}\right| \quad \longleftrightarrow \quad \hat{h}_*^\alpha(\omega) = 2\left|\frac{1+e^{-i\omega}}{2}\right|^{\alpha+1}. \tag{3.10}$$

Thus, our generalized binomial filter $h_+^\alpha(k) = h_-^\alpha(-k)$ is the natural extension of the binomial refinement filter for splines, which plays such a central role in wavelet theory. Interestingly, for $-\frac{1}{2} < \alpha < 0$, although the fractional splines constitute a Riesz basis, their refinement filter does not have the factor $1+e^{-i\omega}$ which is usually believed to be necessary for the construction of unconditional wavelet bases of $\mathbf{L}^2$. We will see that this is not a problem and that these low regularity splines can yield wavelets that are perfectly valid, in spite of their singularities at the integers.

4. Approximation Properties. So far our generalization of splines has proceeded without any major surprises. It is only when we look at their ability to approximate functions that the fractional splines start revealing their less intuitive properties. Here, fractional orders of approximation become possible because we have left the classical framework of the Strang–Fix theory of approximation [18, 34, 6].

4.1. Reproduction of Polynomials. It is well known that the classical B-splines reproduce the polynomial of degree less than or equal to n. What about the fractional splines? It turns out that the noninteger part of α buys us one extra degree.

When we say that a function φ reproduces the polynomials of degree n, we mean that there exist some sequences $c_m(k)$ such that

$$x^m = \sum_{k\in\mathbb{Z}} c_m(k)\varphi(x-k), \quad m = 0,\ldots,n. \tag{4.1}$$

Hence it follows that any polynomial of degree n is expressible as a linear combination of the integer shifts of φ. This polynomial reproduction condition has another equivalent form, which is simpler to work with:

$$\sum_{k\in\mathbb{Z}} (x-k)^m\varphi(x-k) = C_m, \quad m = 0,\ldots,n, \tag{4.2}$$

where the C_m's are some constants. In particular, $C_0 = 1$ if φ satisfies the partition of unity. By using Poisson's summation formula, one gets an equivalent relation in the Fourier domain, the so-called Strang–Fix condition of order $L = n+1$ [33, 32]:

$$\hat{\varphi}(0) = 1 \text{ and } \hat{\varphi}^{(m)}(2k\pi) = 0 \text{ for } \begin{cases} k\in\mathbb{Z}\setminus\{0\}, \\ m = 0,\ldots,n, \end{cases} \tag{4.3}$$

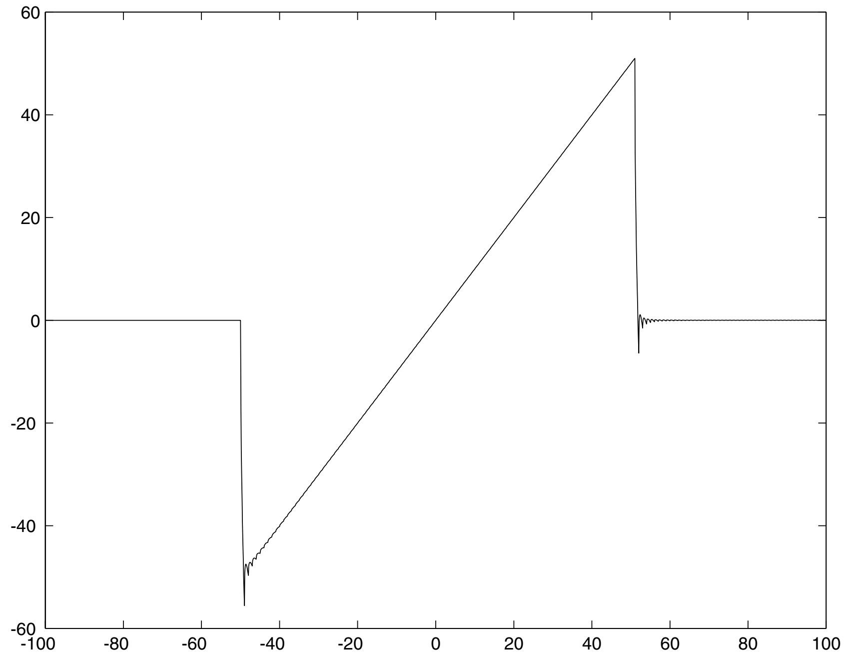

Fig. 4.1 *Plot of* $\sum_{|n|\le 50}(n+\frac{3}{4})\beta_+^{\frac{1}{2}}(x-n)$ *which exemplifies the reconstruction of* $f(x)=x$ *by a linear combination of fractional B-splines of degree one-half.*

where $\hat{\varphi}^{(m)}$ denotes the mth derivative of the Fourier transform of φ. Without any hypothesis on φ other than absolute integrability of f and $x^n\varphi(x)$, the Strang–Fix equivalence is only true in the sense of distributions. For the convergence of (4.2)—and thus of (4.1)—to hold pointwise, we may require that φ decay at least like $|x|^{-m-\varepsilon}$ with $\varepsilon > 0$.

For the fractional B-splines, it is clear that $\hat{\beta}^\alpha(0) = 1$. Moreover, a standard Taylor series argument shows that, for $k \neq 0$, $\hat{\beta}^\alpha(\omega+2k\pi) = C\omega^{\alpha+1}$ as $\omega \to 0$. Thus, $\hat{\varphi}^{(m)}(\omega) = C'(\omega-2k\pi)^{\alpha+1-m}$ as $\omega \to 2k\pi \neq 0$, which means that the fractional B-splines satisfy (4.3) provided that $\alpha+1-m > 0$, i.e., for all $m \le \lceil\alpha\rceil$. The fact that they decay like $|x|^{-\alpha-2}$ (except possibly at the integers when $\alpha < 0$) implies that they reproduce the polynomials of degree $n \le \lceil\alpha\rceil$, pointwise with the exception of the integers when $\alpha < 0$. This makes us jump to the next higher integer $\lceil\alpha\rceil$ when α is non-integer. As an example, it is possible to reproduce the constant, and more surprisingly, the monomial x with a linear combination of shifts of $\sqrt{x_+}$; this is shown in Figure 4.1.

4.2. Fractional Order of Approximation. We now investigate the behavior of the spline approximation error as a function of the scale (or sampling step) a. For this purpose, we define the fractional spline spaces at scale a:

$$\mathbf{S}_a^\alpha = \left\{ s_a : \exists c \in \ell^2, \qquad s_a(x) = \sum_{k\in\mathbb{Z}} c(k)\beta^\alpha\left(\frac{x}{a}-k\right)\right\},$$

which involves stretching the basis functions by a factor of a and spacing them accordingly. Given an arbitrary function $f \in \mathbf{L}^2$, we determine its least-squares approximation in $\mathbf{S}_a^\alpha$ by applying the following orthogonal projection operator (cf. [3]):

$$\mathcal{P}_a f = \sum_{k\in\mathbb{Z}} \left\langle f, \frac{1}{a}\mathring{\beta}^\alpha\left(\frac{\cdot}{a} - k\right)\right\rangle \beta^\alpha\left(\frac{\cdot}{a} - k\right), \tag{4.4}$$

where $\mathring{\beta}^\alpha \in \mathbf{S}^\alpha$ is the dual B-spline whose Fourier transform is

$$\hat{\mathring{\beta}}^\alpha(w) = \frac{\hat{\beta}^\alpha(\omega)}{\sum_{k\in\mathbb{Z}} |\hat{\beta}^\alpha(\omega + 2k\pi)|^2}.$$

Clearly, (4.4) defines a projector because the functions β^α and $\mathring{\beta}^\alpha$ are biorthonormal; i.e., $\langle \beta^\alpha, \beta^\alpha(\cdot - k)\rangle = \delta_k$. The main result of this section is that the approximation error $\|f - \mathcal{P}_a\|_{\mathbf{L}^2}$ decays like $a^{\alpha+1}$ when $a \to 0$. This claim is substantiated with the following error estimates.

Theorem 4.1. *The fractional splines have a fractional order of approximation $\alpha + 1$. Specifically, the least-squares approximation error is bounded by*

$$\forall f \in \mathbf{W}_2^{\alpha+1}, \quad \|f - \mathcal{P}_a\|_{\mathbf{L}^2} \le \frac{\sqrt{2\zeta(\alpha+2) - \frac{1}{2}}}{\pi^{\alpha+1}} \|D^{\alpha+1} f\|_{\mathbf{L}^2}\, a^{\alpha+1}, \tag{4.5}$$

and its asymptotic form is

$$\forall f \in \mathbf{W}_2^{\alpha+1}, \quad \|f - \mathcal{P}_a\|_{\mathbf{L}^2} = \frac{\sqrt{2\zeta(\alpha+2)}}{(2\pi)^{\alpha+1}} \|D^{\alpha+1} f\|_{\mathbf{L}^2}\, a^{\alpha+1} \qquad \text{as } a \to 0. \tag{4.6}$$

Proof. This is very similar to the proofs for integer splines given in [7]. We are within the hypotheses of the main approximation theorem in [6, Theorem 1] since β^α satisfies the Riesz conditions for $\alpha > -\frac{1}{2}$ and since f is at least in $\mathbf{W}_2^{\frac{1}{2}+\varepsilon}$ for $\varepsilon > 0$. Defining $E^\alpha(\omega) = \frac{\sum_{n\ne 0} |\hat{\beta}^\alpha(\omega+2n\pi)|^2}{\sum_n |\hat{\beta}^\alpha(\omega+2n\pi)|^2}$, we use this theorem to show that

$$\begin{aligned} \|f - \mathcal{P}_a\|_{\mathbf{L}^2} \quad \le \quad & \left[\frac{1}{2\pi}\int_{|\omega|\le\frac{\pi}{T}} |\hat{f}(\omega)|^2 E^\alpha(\omega T)\, d\omega\right]^{\frac{1}{2}} \\ & + \frac{\sqrt{\zeta(2\alpha+2)}}{\pi^{\alpha+1}} a^{\alpha+1} \left[\frac{1}{2\pi}\int_{|\omega|\ge\frac{\pi}{T}} |\omega|^{2\alpha+2} |\hat{f}(\omega)|^2 d\omega\right]^{\frac{1}{2}}, \end{aligned} \tag{4.7}$$

where we assume that the function f is sufficiently differentiable for the second term on the right-hand side to be finite. Then, using the Cauchy–Schwarz inequality, we write

$$\|f - \mathcal{P}_a\|_{\mathbf{L}^2} \le \left[\sup_{|\omega|\le\pi} \frac{E^\alpha(\omega)}{|\omega|^{2\alpha+2}} + \frac{\zeta(2\alpha+2)}{\pi^{2\alpha+2}}\right]^{\frac{1}{2}} \|D^{\alpha+1} f\|_{\mathbf{L}^2} a^{\alpha+1}.$$

Finally, applying the same technique as in [7, Theorem 4] for finding an accurate upper bound of $\frac{E^\alpha(\omega)}{|\omega|^{2\alpha+2}}$, we get (4.5).

Using (4.7), it is also possible to see that

$$\|f - \mathcal{P}_a\|_{\mathbf{L}^2} = \left[\frac{1}{2\pi}\int |\hat{f}(\omega)|^2 E^\alpha(a\omega)\, d\omega\right]^{\frac{1}{2}} + o(a^{\alpha+1})$$

if f is at least in $\mathbf{W}_2^{\alpha+1}$, $\alpha > -\frac{1}{2}$. Using the explicit formula of the Fourier transform of a fractional spline, we easily find that $E^\alpha(\omega) = \frac{2\zeta(\alpha+2)}{(2\pi)^{2\alpha+2}}|\omega|^{2\alpha+2} + o(|\omega|^{2\alpha+2})$. Thus, using Lebesgue's dominated convergence theorem, we conclude that

$$\lim_{a\to 0} \frac{\|f - \mathcal{P}_a\|_{\mathbf{L}^2}^2}{a^{2\alpha+2}} = \frac{2\zeta(\alpha+2)}{(2\pi)^{2\alpha+2}}\|D^{\alpha+1}f\|_{\mathbf{L}^2}^2,$$

which is equivalent to (4.6). □

Comments.

(i) Fractional orders of approximation are not very common in approximation theory. We are only aware of two other instances where they have been considered. The first is an $\mathbf{L}^\infty$ bound for the interpolation error on a uniform grid using radial basis functions (cf. [8, Theorem 17]). The second is a general approximation theorem by Jetter [17, Theorem 4.2], which specifies the order in the Fourier domain. This type of result may seem surprising because it appears to go against the Strang–Fix theory of approximation, which states the equivalence between the reproduction of polynomials of degree n (here $n = \lceil\alpha\rceil$) and the approximation order $L = n + 1$, which is one more than the degree. The contradiction that this implies for α noninteger is only apparent because the fractional B-splines do not satisfy some of the hypotheses required by the theory (e.g., compact support). One of the strongest versions of the Strang–Fix equivalence that we know requires two assumptions (cf. [6, Theorem 3], but see also [10]): a Riesz basis condition and some inverse polynomial decay. The fractional B-splines fall short of the second condition by a small margin (cf. Theorem 3.1).

(ii) A generally held belief in wavelet theory is that a first order of approximation would be the minimal requirement for the error to vanish as the scale goes to zero. This limiting behavior is necessary for the representation to be dense in $\mathbf{L}^2$—without it, there are no wavelet bases possible! With fractional splines of degree $-\frac{1}{2} < \alpha < 0$, we have produced examples of functions that have an approximation order smaller than 1 and that still satisfy all the requirements for a multiresolution analysis of $\mathbf{L}^2$. This is obviously only possible because these basis functions are not compactly supported. The other point already mentioned is that the corresponding refinement filters do not have the usual factor $1 + e^{-i\omega}$.

5. Variational Properties. As for the usual splines (of odd degree) [2, 29], the *symmetric* fractional splines are solutions of a variational interpolation problem. This is the generalization of the well-known *minimum curvature* property of the cubic splines. The underlying interpolation problem can be stated as follows: given the uniform samples $\{f(kT)\}$ of a smooth function $f \in \mathbf{L}^2$, find the interpolating function whose samples coincide with those of f and whose derivative of order α has minimal $\mathbf{L}^2$ norm (for $\alpha = 2$, f'' is a good approximation of the curvature). The remarkable result is that the solution to this problem (as well as other related ones) belongs to the vector space

$$\mathbf{S}_{T,*}^{2\alpha-1} = \operatorname*{span}_{k\in\mathbb{Z}} \left\{\beta_*^{2\alpha-1}\left(\frac{x}{T} - k\right)\right\}.$$

In other words, the optimal solution is the fractional spline interpolant f_{int} of f, which is uniquely specified from the sample values $f(kT)$. To show this, we need the following theorem, which generalizes the so-called *first integral relation* for polynomial splines [2].

THEOREM 5.1. *Let $\alpha > \frac{1}{2}$. Then, for all $f \in \mathbf{W}_2^\alpha$, we have*

$$\|D^\alpha f\|_{\mathbf{L}^2}^2 = \|D^\alpha f_{\text{int}}\|_{\mathbf{L}^2}^2 + \|D^\alpha (f - f_{\text{int}})\|_{\mathbf{L}^2}^2, \tag{5.1}$$

where the fractional spline interpolator f_{int} is the unique function of $\mathbf{S}_{T,*}^{2\alpha-1}$ that satisfies $f_{\text{int}}(kT) = f(kT)$ for all integers k.

The proof is given in Appendix C. Of course, if we consider another function $g \in \mathbf{W}_2^\alpha$ that interpolates f at the same knots, then we have the same identity as in Theorem 5.1, with g replacing f. It is thus easy to state the main variational property, which plays a central role in Duchon's theory [16].

COROLLARY 5.2. *Let f belong to the Sobolev space $\mathbf{W}_2^\alpha$. Then the fractional spline of degree $2\alpha - 1$, f_{int}, is the unique function g that minimizes $\|D^\alpha g\|_{\mathbf{L}^2}$ and that interpolates f at its equidistant samples $T\mathbb{Z}$.*

In the same vein, we may consider the following minimization problem:

$$\min_{s \in \mathbf{W}_2^\alpha} \left\{ \sum_{k \in \mathbb{Z}} |f(kT) - s(kT)|^2 + \lambda \|D^\alpha s\|_{\mathbf{L}^2}^2 \right\},$$

where the $f(kT)$'s are our data points and where λ is a given regularization parameter. Here, too, we can show that the optimal solution in $\mathbf{W}_2^\alpha$ is a symmetric fractional spline of degree $2\alpha - 1$; i.e., $s_{\min} \in \mathbf{S}_{T,*}^{2\alpha-1}$. Moreover, as in the usual polynomial spline case [40], the B-spline coefficients of the solution can be computed efficiently by digital filtering. This kind of functional minimization is better suited for fitting noisy data. It allows for a compromise between producing a curve that is close to the data (first error term) and a solution that is reasonably smooth (second regularization term). For an integral value of n, it yields smoothing spline estimators that are widely used in statistics [28, 26, 43, 44].

6. Conclusion. We have extended the family of polynomial splines to fractional orders. We constructed the fractional B-splines by taking fractional finite differences of one-sided power functions x_+^α. What is remarkable is that these new functions inherit all the nice properties of the polynomial B-splines with two exceptions: positivity and compact support. They provide the same ease for dealing with fractional derivatives as the conventional splines do for derivatives. They have simple explicit formulae in both the time and frequency domains. They also generate Riesz bases and satisfy a two-scale relation. Their most notable feature is their order of approximation, namely $\alpha + 1$, which is no longer an integer. These new functions may be used to construct new fractional wavelet bases of $\mathbf{L}^2$ using any of the techniques developed with polynomial splines. For instance, we can readily specify an enlarged family of orthogonal Battle–Lemarié wavelets with a continuous order indexing rather than a discrete one.

The fractional splines interpolate the polynomial splines in the same sense as the gamma function interpolates the factorial—this is more than an analogy, because the gamma function is intimately involved in the definition. By simply varying the fractional exponent α, we have direct control of the most important properties of this family of functions (Hölder continuity, Sobolev regularity, decay, order of approximation, etc.).

Finally, we believe that it is possible to extend most of these results to a nonuniform grid, similar to what has been done with polynomial splines and radial basis functions.

Appendix A. Proof of Lemma 2.3. By definition, the $\left|{r \atop n}\right|$'s are the coefficients of the development of $|1+e^{2i\pi\nu}|^r$ in series of $e^{2ni\pi\nu}$, which means that

$$\left|{r \atop n}\right| = 2^r \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos^r(\pi\nu) e^{-2ni\pi\nu}\, d\nu. \tag{A.1}$$

From this, one gets

$$\begin{aligned}
\left|{r \atop n}\right| - \left|{r \atop n+1}\right| &= 2^{r+1} i \int_{-\frac{1}{2}}^{\frac{1}{2}} \cos^r(\pi\nu)\sin(\pi\nu) e^{-(2n+1)i\pi\nu}\, d\nu \\
&= 2^{r+1}\frac{2n+1}{r+1}\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos^{r+1}(\pi\nu) e^{-(2n+1)i\pi\nu}\, d\nu \quad \text{(by parts)} \\
&= 2^{r}\frac{2n+1}{r+1}\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos^{r}(\pi\nu)(1+e^{-2i\pi\nu}) e^{-2ni\pi\nu}\, d\nu \\
&= \frac{2n+1}{r+1}\left(\left|{r \atop n}\right| + \left|{r \atop n+1}\right|\right), \qquad \text{(A.2)}
\end{aligned}$$

which yields the induction relation

$$\left|{r \atop n+1}\right| = \frac{r-2n+2}{r+2n}\left|{r \atop n}\right|. \tag{A.3}$$

We notice that this equation is exactly the recursion followed by $\binom{r}{n+\frac{r}{2}}$. This implies that $\left|{r \atop n}\right| = \left(\left|{r \atop 0}\right| / \binom{r}{\frac{r}{2}}\right)\binom{r}{n+\frac{r}{2}}$. Moreover, $\left|{r \atop 0}\right|$ is an Euler beta function as shown here:

$$\begin{aligned}
\left|{r \atop 0}\right| &= \frac{2^{r+1}}{\pi}\int_0^{\frac{\pi}{2}} \cos^r(x)\, dx \\
&= \frac{2^r}{\pi} B\left(\frac{1}{2}, \frac{r+1}{2}\right) \\
&= \frac{2^r}{\pi}\frac{\Gamma(\frac{r+1}{2})\Gamma(\frac{1}{2})}{\Gamma(1+\frac{r}{2})} \\
&= \frac{2^r}{\sqrt{\pi}}\frac{\Gamma(\frac{1+r}{2})}{\Gamma(1+\frac{r}{2})} \\
&= 2\frac{\Gamma(r)}{\Gamma(\frac{r}{2})\Gamma(1+\frac{r}{2})} \quad \text{(using Gauss's duplication formula [1])} \\
&= \binom{r}{\frac{r}{2}}, \qquad \text{(A.4)}
\end{aligned}$$

which thus provides the final expression of $\left|{r \atop n}\right|$ under the form (2.6). ☐

Appendix B. Proof of Theorem 3.1. We will proceed in three steps. First, we consider $\alpha > 0$ and show that $|\beta_+^\alpha| \leq C|x|^{-\alpha-2}$. Second, using the same method, we prove the same kind of result for β_*^α for any $\alpha > 0$. Third, we apply the spline induction relations (2.3) and (2.12) to extend this result to negative values of α. Note that, without loss of generality, we can consider in our calculations that $x > 0$.

First Step: Causal B-Splines and $\boldsymbol{\alpha > 0}$. Using (3.1) and integrating N times by parts $2\pi\beta_+^\alpha(x) = \int \hat{\beta}_+^\alpha(\omega)e^{ix\omega}\,d\omega$ yields

$$2\pi(-ix)^N\beta_+^\alpha(x) = \int D^N\hat{\beta}_+^\alpha(\omega)e^{ix\omega}\,d\omega.$$

We are allowed to do these integrations by parts only as long as $\alpha + 1 - N > -1$; for higher values of N, the singularities at $\omega = 2n\pi$, $n \neq 0$, become nonintegrable. For $\alpha > 0$, the integrated terms cancel at infinity. Thus, we take $N = \lceil\alpha\rceil + 1$.

In the neighborhood of $\omega = 2n\pi$ with $n \neq 0$, $\hat{\beta}_+^\alpha(\omega) = (\omega/2n\pi - 1)^{\alpha+1} + O\big((\omega - 2n\pi)^{\alpha+2}\big)$; thus

$$D^N\hat{\beta}_+^\alpha(\omega) = \underbrace{D^N\left(\frac{\omega}{2n\pi} - 1\right)^{\alpha+1}}_{\hat{u}_{n,\alpha}} + O\big((\omega - 2n\pi)^{\alpha+2-N}\big).$$

This defines a function $u_{n,\alpha}$ through its Fourier transform, and we have

$$u_{n,\alpha}(x) = (-i)^N\Gamma(\alpha+2)\frac{\sin\pi\alpha}{\pi}\frac{e^{2ni\pi x}}{(2ni\pi)^{\alpha+1}}x^{N-\alpha-2}$$

for $n \geq 1$ and $u_{n,\alpha}(x) = 0$ for $n < 0$; this is valid only for positive values of x. We now consider the function $u_\alpha(x)$ defined by

$$u_\alpha(x) = \sum_{n\geq 1} u_{n,\alpha}(x), \tag{B.1}$$

which is a uniformly convergent series since $\alpha > 0$. u_α is such that its Fourier transform satisfies $\hat{u}_\alpha(\omega) = D^N\hat{\beta}_+^\alpha(\omega) + O((\omega - 2n\pi)^{\alpha+2-N})$. Thus, integrating by parts once again, we have

$$(-ix)^N\beta_+^\alpha(x) - u_\alpha(x) = -\frac{1}{2i\pi x}\int D\left\{D^N\hat{\beta}_+^\alpha(\omega) - \hat{u}_\alpha(\omega)\right\}e^{ix\omega}\,d\omega. \tag{B.2}$$

The final task is to show that the expression within the integral sign is absolutely integrable. If this is the case, then we can conclude that the right-hand side of (B.2) is $o(x^{-1})$; this is the consequence of a standard theorem in integration theory which states that if $f \in \mathbf{L}^1$, then $\int f(x)e^{-i\omega x}\,dx$ tends to zero as $x \to +\infty$.

In order to prove that the integrand on the right-hand-side of (B.2) is in $\mathbf{L}^1$, we decompose the integral into a sum of definite integrals $\sum_{n\in\mathbb{Z}}\int_{(2n-1)\pi}^{(2n+1)\pi}[\ldots]d\omega$. We can assume that ω is positive, since the integrand of (B.2) is conjugate symmetric when ω is changed into $-\omega$. We can thus also assume that $n \geq 0$. The case $n = 0$ can be dealt with easily since the integrand is locally integrable (which implies that its integral tends to zero as $x \to \infty$). We thus concentrate on the case $n \geq 1$. According to our definition of the complex fractional power, we get $\hat{\beta}_+^\alpha(\omega) = \frac{(1-e^{-i\omega})^{\alpha+1}}{(i\omega)^{\alpha+1}} = \frac{\hat{\beta}_+^\alpha(\omega-2n\pi)}{\omega^{\alpha+1}}(\omega - 2n\pi)^{\alpha+1}$. Using Leibnitz's chain rule for differentiation, we have

$$D\left\{D^N\hat{\beta}_+^\alpha(\omega)-\hat{u}_\alpha(\omega)\right\} = \underbrace{-\sum_{k\in\mathbb{N}\setminus\{0,n\}} D\hat{u}_{k,\alpha}(\omega)}_{T_1(\omega)}$$
$$+\underbrace{\sum_{k=1}^{N+1}\binom{N+1}{k}(-1)^k D^k\left\{\frac{\hat{\beta}_+^\alpha(\omega-2n\pi)}{\omega^{\alpha+1}}\right\}D^{N+1-k}(\omega-2n\pi)^{\alpha+1}}_{T_2(\omega)}$$
$$+\underbrace{\left(\frac{\hat{\beta}_+^\alpha(\omega-2n\pi)}{\omega^{\alpha+1}}-\frac{\hat{\beta}_+^\alpha(0)}{(2n\pi)^{\alpha+1}}\right)D^{N+1}(\omega-2n\pi)^{\alpha+1}}_{T_3(\omega)},$$

the modulus of which is to be integrated over $[(2n-1)\pi,(2n+1)\pi]$ and summed for $n\geq 1$. Using the triangle inequality, we consider the moduli of the three terms $T_i(\omega)$. We denote by E_1,E_2,E_3 the sum of the contributions over $n\geq 1$ of T_1,T_2, and T_3, respectively. If these are finite, we can deduce that

$$\int\left|D\left\{D^N\hat{\beta}_+^\alpha(\omega)-\hat{u}_\alpha(\omega)\right\}\right|d\omega \leq E_1+E_2+E_3<\infty,$$

which proves our claim. We now show how these terms can be bounded.

- For the first one, we have

$$\begin{aligned}E_1 &= \sum_{n\geq 1}\int_{(2n-1)\pi}^{(2n+1)\pi}|T_1(\omega)|\,d\omega\\ &\leq \sum_{n\geq 1}\sum_{k\in\mathbb{N}\setminus\{0,n\}}\int_{(2n-1)\pi}^{(2n+1)\pi}\frac{C}{(2k\pi)^{\alpha+1}}|\omega-2k\pi|^{\alpha-N}\,d\omega\\ &\leq \sum_{k\geq 1}\sum_{n\in\mathbb{N}\setminus\{0,k\}}\int_{(2n-1)\pi}^{(2n+1)\pi}\frac{C}{(2k\pi)^{\alpha+1}}|\omega-2k\pi|^{\alpha-N}\,d\omega \text{ (Fubini)}\\ &\leq \frac{2C\pi^{\alpha-N+1}}{\alpha-N+1}\sum_{k\geq 1}\frac{1}{(2k\pi)^{\alpha+1}}<\infty,\end{aligned}$$

  where C is a constant that does not depend on k,n.
- We observe that $D^k\{\omega^{-\alpha-1}\hat{\beta}_+^\alpha(\omega-2n\pi)\}$ can be roughly bounded by $\frac{C'}{n^{\alpha+1}}$ within the interval $\omega\in[(2n-1)\pi,(2n+1)\pi]$. Moreover, for $k\geq 1$, the expression $|D^{N+1-k}(\omega-2n\pi)^{\alpha+1}|$ can be integrated over this same interval and is bounded by some constant C''. Finally, the summation for k between 1 and $N+1$ yields $\int_{(2n-1)\pi}^{(2n+1)\pi}|T_2(\omega)|\,d\omega\leq\frac{C'''}{n^{\alpha+1}}$. This term can be finitely summed for $n\geq 1$ and thus $E_2<\infty$.
- We rewrite T_3 as

$$\begin{aligned}T_3(\omega) &= \left(\frac{\hat{\beta}_+^\alpha(\omega-2n\pi)-\hat{\beta}_+^\alpha(0)}{\omega-2n\pi}-\frac{\hat{\beta}_+^\alpha(0)}{(2n\pi)^{\alpha+1}}\frac{\omega^{\alpha+1}-(2n\pi)^{\alpha+1}}{\omega-2n\pi}\right)\\ &\quad\times\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+1-N)}\frac{(\omega-2n\pi)^{\alpha-N+1}}{\omega^{\alpha+1}}.\end{aligned}$$

  In the interval $\omega\in[(2n-1)\pi,(2n+1)\pi]$, this term is simply bounded by $\frac{K}{n^{\alpha+1}}|\omega-2n\pi|^{\alpha-N+1}$, where K is a constant that does not depend on n. Since

$|\omega - 2n\pi|^{\alpha-N+1}$ is locally integrable, we thus have $E_3 \leq \sum_{n\geq 1} \frac{K'}{n^{\alpha+1}}$, which is finite.

This proves that

$$(-ix)^N \beta_+^\alpha(x) - u_\alpha(x) = o(x^{-1}).$$

Finally, since $u_\alpha(x) \propto x^{N-\alpha-2}$, we find that $\beta_+^\alpha(x) \propto x^{-\alpha-2}$.

Second Step: Symmetrized Splines and $\boldsymbol{\alpha > 0}$. We need not consider $\alpha \in 2\mathbb{N}+1$, since this corresponds to standard, compactly supported B-splines. We follow the same steps as above. In particular, we again do $N = \lceil \alpha \rceil + 1$ integrations by part, which provides us with an integral expression for $(-ix)^N \beta_*^\alpha(x)$. Analogously to u_α, we define

$$D^N \hat{\beta}_+^\alpha(\omega) = \underbrace{D^N \left| \frac{\omega}{2n\pi} - 1 \right|^{\alpha+1}}_{\hat{v}_{n,\alpha}} + O\big((\omega - 2n\pi)^{\alpha+2-N}\big),$$

which yields

$$v_{n,\alpha}(x) = -(-i)^N \Gamma(\alpha+2) \frac{\cos(\frac{\pi}{2}\alpha)}{\pi} \frac{e^{2ni\pi x}}{|2n\pi|^{\alpha+1}} x^{N-\alpha-2}$$

for all $n \neq 0$. As for (B.1), we define a function v_α that is the sum over $n \in \mathbb{Z} \setminus \{0\}$ of the $v_{n,\alpha}$. Subtracting this function from $(-ix)^N \beta_*^\alpha(x)$ allows an additional integration by parts, whose integrand can be shown to be in $\mathbf{L}^1$ (same tedious proof as for β_+^α). Hence, we conclude that

$$(-ix)^N \beta_*^\alpha(x) - v_\alpha(x) = o(x^{-1}),$$

which implies that the decay of β_*^α is proportional to $x^{-\alpha-2}$ as $x \to \infty$.

Third Step: $\boldsymbol{\alpha \leq 0}$. We shall first prove the following technical lemma.

LEMMA B.1. *Let r, s, d be three real numbers and $f(x), g(x)$ two functions that are related through*

$$g(x) = xf(x) + (r-x)f(x-1). \tag{B.3}$$

If there exist constants a_0, a_1, a_2 such that

$$\forall x \in \mathbb{R}_+, \quad |g(x)| \leq \frac{a_0\{[x-d]\}_*^s + a_1}{1+x^{r+1}},$$
$$\forall x \in [0,1[, \quad |f(x)| \leq a_2\{[x-d]\}_*^s,$$

then we have the following upper bound for $f(x)$:

$$\forall x \in \mathbb{R}, \quad |f(x)| \leq \frac{a_0'\{[x-d]\}_*^s + a_1'}{1+x^r}.$$

Proof. We define $u(x) = \frac{\Gamma(x+1)}{\Gamma(x-r+1)}$ so that we can write (B.3) as $x^{-1}u(x)g(x) = u(x)f(x) - u(x-1)f(x-1)$. This equation can be inverted in the following way:

$$u(x)f(x) = u(x - \lfloor x \rfloor)f(x - \lfloor x \rfloor) + \sum_{k=0}^{\lfloor x \rfloor - 1} \frac{u(x-k)}{x-k} g(x-k).$$

Using Stirling's formula, we show that there exist constants b_0, b_1 such that $b_0(1+x^r) \leq |u(x)| \leq b_1(1+x^r)$ for all positive x. This implies that the first term on the right-hand side of the above equation is upper bounded by Const$\times\{[x-d]\}_*^s$, while the summation term is convergent as $\sum_{k\geq 1} k^{-2}$ for all positive x and is thus bounded by Const$\times\{[x-d]\}_*^s$+Const$'$. Finally, by applying the inequality $|u(x)|^{-1} \leq b_0^{-1}/(1+x^r)$ we get the upper bound for $f(x)$. ◻

We apply this lemma to $\beta_+^\alpha(x)$, which is known to satisfy an induction relation similar to (B.3), namely, (2.3). In that case we identify $d = 0$, $r = \alpha+2$, $g(x) = \beta_+^{\alpha+1}(x)$, and $f(x) = \frac{1}{\alpha+1}\beta_+^\alpha(x)$, where we assume $-1 < \alpha < 0$ (of course, the case $\alpha = 0$ is already solved since β_+^0 is compactly supported). We easily verify that $|\beta_+^\alpha(x)| \leq a_2\{[x]\}_*^\alpha$ for $x \in [0,1[$ so that we can take $s = \alpha$. Finally, thanks to the proof of the decay for fractional splines of positive degree, we observe that the condition on $g(x)$ is satisfied as well. This proves the claim of Theorem 3.1 for $\alpha < 0$ and $\beta^\alpha = \beta_+^\alpha$.

In order to apply our lemma to β_*^α, we need to define the intermediary function $\gamma^{\alpha+1}(x)$:

$$\gamma^{\alpha+1}(x) = \frac{x}{\alpha+1}\beta_*^\alpha\left(x - \frac{\alpha+1}{2}\right) + \frac{\alpha+2-x}{\alpha+1}\beta_*^\alpha\left(x - 1 - \frac{\alpha+1}{2}\right).$$

Thanks to (2.12) it can be verified that we also have

$$\beta_*^{\alpha+2}\left(x - \frac{\alpha+3}{2}\right) = \frac{x}{\alpha+2}\gamma^{\alpha+1}(x) + \frac{\alpha+3-x}{\alpha+2}\gamma^{\alpha+1}(x-1).$$

Then, we apply the lemma twice: First, we identify $r = \alpha+3$, $g = \beta_*^{\alpha+2}$, and $f = \frac{1}{\alpha+2}\gamma^{\alpha+1}$, where we assume $-1 < \alpha \leq 0$. Using the definition of $\gamma^{\alpha+1}$ we see that we also have $|\gamma^{\alpha+1}(x)| \leq a_2\{[x - \frac{\alpha+1}{2}]\}_*^\alpha$ for $x \in [0,1[$, which makes us identify $d = \frac{\alpha+1}{2}$ and $s = \alpha$. Since we already know the rate of decay of $g(x)$, i.e., $x^{-\alpha-4}$, we thus claim that

$$|\gamma^{\alpha+1}(x)| \leq \frac{a_0'\{[x - \frac{\alpha+1}{2}]\}_*^\alpha + a_1'}{1 + x^{\alpha+3}}$$

for all positive x.

Second, we set $r = \alpha+2$, $g(x) = \gamma^{\alpha+1}(x)$, and $f(x) = \frac{1}{\alpha+1}\beta_*^\alpha(x - \frac{\alpha+1}{2})$. We obviously have $|f(x)| \leq a_2'\{[x - \frac{\alpha+1}{2}]\}_*^\alpha$ for $x \in [0,1[$, which makes us identify $d = \frac{\alpha+1}{2}$ and $s = \alpha$ in the above lemma. We thus conclude that

$$\left|\beta_*^\alpha\left(x - \frac{\alpha+1}{2}\right)\right| \leq \frac{a_0''\{[x - \frac{\alpha+1}{2}]\}_*^\alpha + a_1''}{1 + x^{\alpha+2}}$$

for positive x. This is equivalent to the form of (3.4) for $\beta^\alpha = \beta_*^\alpha$. ◻

Appendix C. Proof of Theorem 5.1. The condition $f \in \mathbf{W}_2^\alpha$ with $\alpha > \frac{1}{2}$ guarantees that $f(T\mathbb{Z}) \in \ell^2$ [6]. A consequence is that the 2π-periodic function $g(\theta) = \sum_{n\in\mathbb{Z}} f(nT)e^{ik\theta}$ is in $\mathbf{L}^2[0,2\pi]$. If there exists an $\mathbf{L}^2$ function $f_{\text{int}} = \sum_{k\in\mathbb{Z}} c_k \beta_*^{2\alpha-1}(\frac{\cdot}{T} - k)$ such that $f_{\text{int}}(nT) = f(nT)$ for all integers n, then the coefficients c_k must belong to ℓ^2 (lower Riesz condition). Thus the interpolation condition can be rewritten as

$$g(\theta) = \left(\sum_k c_k e^{ik\theta}\right)\underbrace{\left(\sum_k \beta_*^{2\alpha-1}(k)e^{ik\theta}\right)}_{A(\theta)}.$$

Using Poisson's formula (valid pointwise since $\beta_*^{2\alpha-1}$ and $\hat{\beta}_*^{2\alpha-1}$ decay faster than $|x|^{-1-\varepsilon}$ and $|\omega|^{-1-\varepsilon}$, and since $\hat{\beta}_*^{2\alpha-1}$ is continuous) we have $A(\theta) = \sum_k \hat{\beta}_*^{2\alpha-1}(\theta - 2k\pi)$, which is obviously always strictly greater than a positive constant, namely, $A^2_{\alpha-1}$, in (3.8). This implies that $g(\theta)A(\theta)^{-1}$ is in $\mathbf{L}^2[0, 2\pi]$. Thus, $\{c_k\}_{k\in\mathbb{Z}}$ exists and is given uniquely by $c_k = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)A(\theta)^{-1}e^{-ik\theta}d\theta$. This proves the existence and the unicity of the interpolator in $\mathbf{S}^{2\alpha-1}_{T,*}$.

Next, we observe that the condition $\alpha > \frac{1}{2}$ implies that any function $\varphi \in \mathbf{S}^{2\alpha-1}_{T,*}$ belongs to $\mathbf{W}_2^\alpha$ as well. Moreover, if $d(x)$ is a function of $\mathbf{W}_2^\alpha$ such that $d(n) = 0$ for all $n \in \mathbb{Z}$, then

$$\sum_n \hat{d}(\omega + 2n\pi) = 0 \quad \text{for almost every } \omega \in [0, 2\pi]. \tag{C.1}$$

The proof of this claim comes from the fact that $\sum_n \hat{d}(\omega + 2n\pi)$ is in $\mathbf{L}^2[0, 2\pi]$ as a consequence of the fact that $\alpha > \frac{1}{2}$ (this can be seen by using a two-step bounding process which involves Minkowsky and Cauchy–Schwarz inequalities). Thus, we can apply Fourier's theorem about the decomposition of 2π-periodic $\mathbf{L}^2[0, 2\pi]$ functions into sinusoids $e^{in\omega}$. (Note that this is another flavor of Poisson's summation formula.)

Finally, let $\hat{f}_{\text{int}}(\omega) = C(\omega T)\hat{\beta}_*^{2\alpha-1}(\omega T)$, where $C(\omega T)$ is 2π-periodic and is in $\mathbf{L}^2[0, 2\pi]$. If $f \in \mathbf{W}_2^\alpha$ then $d(x) = f(Tx) - f_{\text{int}}(Tx)$ is in $\mathbf{W}_2^\alpha$ as well, and satisfies $d(n) = 0$ for all $n \in \mathbb{Z}$. We have

$$\begin{aligned}
\langle D^\alpha \{f - f_{\text{int}}\}, D^\alpha f_{\text{int}} \rangle &= \frac{T}{2\pi}\int \overline{\hat{d}(T\omega)}\, \hat{f}_{\text{int}}(\omega)|\omega|^{2\alpha}\, d\omega \\
&= \frac{2^{2\alpha}}{2\pi T^{2\alpha}} \int \overline{d(\omega)}\, C(\omega) \left|\sin\frac{\omega}{2}\right|^{2\alpha} d\omega \\
&= \frac{2^{2\alpha}}{2\pi T^{2\alpha}} \sum_{n\in\mathbb{Z}} \int_0^{2\pi} \overline{d(\omega + 2n\pi)}\, C(\omega) \left|\sin\frac{\omega}{2}\right|^{2\alpha} d\omega \\
&= \frac{2^{2\alpha}}{2\pi T^{2\alpha}} \int_0^{2\pi} \overline{\sum_{n\in\mathbb{Z}} d(\omega + 2n\pi)}\, C(\omega) \left|\sin\frac{\omega}{2}\right|^{2\alpha} d\omega \\
&= 0,
\end{aligned}$$

where the exchange of the $\int$ and $\sum$ signs is justified by the uniform convergence of the expression (Fubini's theorem). Equation (5.1) is then a simple consequence of the decomposition of the $\mathbf{L}^2$ norm

$$\|D^\alpha f\|^2_{\mathbf{L}^2} = \|D^\alpha\{f - f_{\text{int}}\}\|^2_{\mathbf{L}^2} + 2\langle D^\alpha \{f - f_{\text{int}}\}, D^\alpha f_{\text{int}}\rangle + \|D^\alpha f_{\text{int}}\|^2_{\mathbf{L}^2}. \qquad \square$$

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