On sampling lattices with similarity scaling relationships

Steven Bergner [(1)], Dimitri Van De Ville[(2)], Thierry Blu[(3)], and Torsten Möller[(1)]

(1) GrUVi-Lab, Simon Fraser University, Burnaby, Canada.
(2) BIG, Ecole Polytechnique Fédérale de Lausanne, Switzerland.
(3) The Chinese University of Hong Kong, Hong Kong, China.
sbergner@cs.sfu.ca, thierry.blu@m4x.org, Dimitri.VanDeVille@epfl.ch, torsten@cs.sfu.ca

Abstract:

We provide a method for constructing regular sampling lattices in arbitrary dimensions together with an integer dilation matrix. Sub-sampling using this matrix leads to a similarity-transformed version of the lattice with a chosen density reduction. These lattices are interesting candidates for multi-dimensional wavelet constructions with a limited number of sub-bands.

1. Introduction to sampling lattices and related work

A sampling lattice is a set of points $\{\mathbf{Rk} : \mathbf{k} \in \mathbb{Z}^n\} \subset \mathbb{R}^n$ that is closed under addition and inversion. The non-singular generating matrix $\mathbf{R} \in \mathbb{R}^{n\times n}$ contains basis vectors in its columns. Lattice points are uniquely indexed by $\mathbf{k} \in \mathbb{Z}^n$ and the neighbourhoods around all sampling points are identical. This makes them suitable sampling patterns for the reconstruction of shift-invariant spaces.

Sub-sampling schemes for lattices are expressed in terms of a dilation matrix $\mathbf{K} \in \mathbb{Z}^{n\times n}$ forming a new lattice with generating matrix $\mathbf{RK}$. The reduction rate in sampling density corresponds to

$$\det \mathbf{K} = \alpha^n = \delta \in \mathbb{Z}^+. \tag{1}$$

Dyadic sub-sampling discards every second sample along each of the n dimensions resulting in a $\delta = 2^n$ reduction rate. To allow for fine-grained scale transitions we are particularly interested in low sub-sampling rates, such as $\delta = 2$ or 3.

As discussed by van de Ville et al. [8] the 2D quincunx sub-sampling is an interesting case permitting a two-scale relation. With the implicit assumption of only considering subsets of the Cartesian lattice it is shown that a similarity two-channel dilation may not extend for $n > 2$.

We show that by permitting more general basis vectors in $\mathbb{R}^n$ the desired fixed-rate dilation becomes possible for any n. Our construction produces a variety of lattices making it possible to include additional quality criteria into the search as they may be computed from the Voronoi cell of the lattice [9] including packing density and expected quadratic quantization error (second order moment). Agrell et al. [1] improve efficiency for the computation by extracting Voronoi relevant neighbours. Another sampling quality criterion appears in the work of Lu et al. [4] in form of an analytic alias-free sampling condition that is employed in a lattice search.

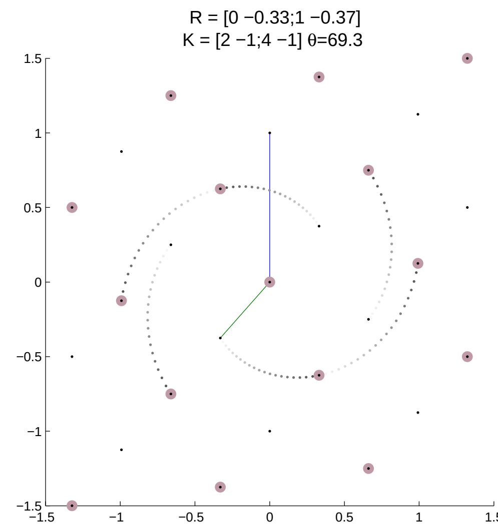

Figure 1: 2D lattice with basis vectors and sub-sampling as given by $\mathbf{R}$ and $\mathbf{K}$ in the diagram title. The spiral shaped points correspond to a sequence of fractional sub-samplings $\mathbf{RK}^s$ for $s = 0..1$ with the notable feature that for $s = 1$ one obtains a subset of the original lattice sites shown as thick dots. This repeats for any further integer power of $\mathbf{K}$, each time reducing the sample density by $|\det \mathbf{K}| = 2$.

2. Lattice construction

We are looking for a non-singular lattice generating matrix $\mathbf{R}$ that, when sub-sampled by a dilation matrix $\mathbf{K}$ with reduction rate $\delta = \alpha^n$, results in a similarity-transformed version of the same lattice, that is, it can be scaled and rotated by a matrix $\mathbf{Q}$ with $\mathbf{Q}^T\mathbf{Q} = \alpha^2\mathbf{I}$. An illustration of a sub-sampling resulting in a rotation by $\theta = \arccos \frac{1}{2\sqrt{2}}$ is given in Figure 1. Formally, this kind of relationship can be expressed as

$$\mathbf{QR} = \mathbf{RK} \tag{2}$$

leading to the observation that sub-sampling $\mathbf{K}$ and scaled rotation $\mathbf{Q}$ are related by a similarity transform

$$\mathbf{R}^{-1}\mathbf{QR} = \mathbf{K}. \tag{3}$$

Using a matrix $\mathbf{J}_2 = \begin{bmatrix} 1 & j \\ 1 & -j \end{bmatrix}$ it is possible to diagonalize a 2D rotation matrix by the following similarity transform

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \mathbf{J}_2^{-1} \begin{bmatrix} e^{j\theta} & 0 \\ 0 & e^{-j\theta} \end{bmatrix} \mathbf{J}_2 = \mathbf{J}_2^{-1} \mathbf{\Delta} \mathbf{J}_2. \tag{4}$$

Using this observation to replace the scaled rotation matrix $\mathbf{Q}$ in Equation 3 leads to

$$\begin{aligned} \mathbf{K} &= \mathbf{R}^{-1}\mathbf{Q}\mathbf{R} \\ \mathbf{K} &= \alpha \mathbf{R}^{-1}\mathbf{J}_n^{-1}\mathbf{S}\mathbf{\Delta}\mathbf{S}^{-1}\mathbf{J}_n\mathbf{R} \\ \mathbf{K} &= \alpha \mathbf{P}\mathbf{\Delta}\mathbf{P}^{-1} \end{aligned} \tag{5}$$

with

$$\begin{aligned} \mathbf{R} &= \mathbf{J}_n^{-1}\mathbf{S}\mathbf{P}^{-1} \\ \mathbf{Q} &= \alpha \mathbf{J}_n^{-1}\mathbf{\Delta}\mathbf{J}_n. \end{aligned} \tag{6}$$

Thus, given a matrix $\mathbf{K}$ that has an eigen-decomposition corresponding to that of a uniformly scaled rotation matrix, we can compute the lattice generating matrix $\mathbf{R}$ as in Equation 6. The elements of the diagonal matrix $\mathbf{S}$ inserted in the construction of $\mathbf{R}$ scale the otherwise unit eigenvectors in the columns of $\mathbf{P}$. Below, we will refer to this construction as function formRQ($\mathbf{K}, \mathbf{S}$) using $\mathbf{S} = \mathbf{I}$ by default.

2.1 Constructing suitable dilation matrices $\mathbf{K}$

The eigenvalues of $\mathbf{K}, \mathbf{\Delta}$ and $\mathbf{Q}$ impose restrictions on their shared characteristic polynomial $d(\lambda) = \det(\mathbf{K} - \lambda\mathbf{I}) = \sum_{k=0}^{n} c_k \lambda^k$ as discussed in the appendix. For the case $n = even$ with the only non-zero coefficients $c_0 = \delta,\ c_{n/2}^2 < 4\delta,\ c_n = 1$ this leaves a finite number of different options for $c_{n/2}$. The case $n = odd$ permits a single possible polynomial with non-zero coefficients $c_0 = -\delta,\ c_n = 1$. For these monic polynomials it is possible to directly construct a candidate $\mathbf{K}$ via the companion matrix ([6], p. 192)

$$\mathbf{K} = \begin{bmatrix} 0 & & & & -c_0 \\ 1 & 0 & & & -c_1 \\ & 1 & 0 & & \vdots \\ & & \ddots & \ddots & -c_{n-2} \\ & & & 1 & -c_{n-1} \end{bmatrix}. \tag{7}$$

This allows to construct a lattice fulfilling the self-similar sub-sampling condition for any dimensionality n, one for every possible characteristic polynomial.

With this starting point it is possible to construct additional suitable dilation matrices via a similarity transform with a unimodular matrix $\mathbf{T}$

$$\mathbf{K}_T = \mathbf{T}\mathbf{K}\mathbf{T}^{-1} = \mathbf{P}_T \mathbf{\Delta} \mathbf{P}_T^{-1}. \tag{8}$$

Using a unimodular rather than any non-singular $\mathbf{T}$ guarantees that $\mathbf{T}^{-1}$ is also unimodular following from the fact that $\mathbf{T}^{-1}$ can be constructed from the adjugate (the transposed co-factor matrix) of $\mathbf{T}$. Thus, $\mathbf{K}_T$ remains an integer matrix by this transform. Possible generators for this unimodular group are discussed in ([5], pp. 23). Our implementation, referred to as function genUnimodular(n), uses a construction of $\mathbf{T} = \mathbf{L}\mathbf{U}$ from several random integer lower and upper triangular matrices having ones on their diagonal.

It is not guaranteed that all possible $\mathbf{K}$ for a given characteristic polynomial can be generated through a similarity transform with some $\mathbf{T}$. However, formRQ($\mathbf{K}_T$) provides numerous non-equivalent $\mathbf{R}_T$ lattice generators. Among them it is possible to apply further criteria to select the "best" lattice.

An alternative to transforming $\mathbf{K}$ is the eigenvector scaling by diagonal matrix $\mathbf{S}$ in Equation 6. Using non-unit scaling allows to produce further lattices for any given $\mathbf{K}$ resulting in an n-dimensional continuous search space.

2.2 Construction Algorithm

The steps for constructing lattices with the desired sub-sampling matrices are summarized in algorithm 1. The function compoly(n, α, C) is defined in the appendix. A possible implementation for the function genUnimodular(n) is described in Section 2.1 and formRQ($\mathbf{K}$) is defined near Equation 6.

Algorithm 1 genLattices(n, δ)

Llist $\leftarrow \{\}$
Ks $\leftarrow$ genKompans(n, δ)
Ts $\leftarrow$ genUnimodular(n) $\cup \{\mathbf{I}\}$
for all $\mathbf{K} \in$ Ks **do**
  for all $\mathbf{T} \in$ Ts **do**
    $\mathbf{K}_T = \mathbf{T}\mathbf{K}\mathbf{T}^{-1}$
    $(\mathbf{R}_T, \mathbf{Q}_T) \leftarrow$ formRQ($\mathbf{K}_T$)
    Llist$\leftarrow$ Llist$\cup\{(\mathbf{K}_T, \mathbf{R}_T, \mathbf{Q}_T)\}$
  end for
end for
return Llist

Algorithm 2 genKompans(n, δ)

Ks = $\{\}$
if n is even **then**
  for all $C \in \mathbb{Z} : C^2 < 4\delta$ **do**
    Ks $\leftarrow$ Ks $\cup$ compoly($n, \delta^{\frac{1}{n}}, C$)
  end for
else $\{n$ is odd$\}$
  Ks $\leftarrow \{$compoly($n, \delta^{\frac{1}{n}}$)$\}$
end if
return Ks

It should be noted that the list of lattices returned by genLattices may contain several equivalent copies of the same lattice. A Gram matrix implicitly represents angles between basis vectors as $\mathbf{A} = \mathbf{R}^T\mathbf{R}$. Two lattices $\mathbf{R}_1$ and $\mathbf{R}_2$, scaled to same determinant, are equivalent if their Gram matrices are related via $\mathbf{A}_1 = \mathbf{T}^T\mathbf{A}_2\mathbf{T}$ with a unimodular matrix $\mathbf{T} \in \mathbb{Z}^{n\times n}$ and $|\det \mathbf{T}| = 1$. Determining this unimodular matrix is known to be a difficult problem, as it for instance also occurs when relating the adjacency matrices of two supposedly isomorphic graphs. Hence, our current method employs a simpler necessary test for equivalence by comparing the first few elements of the set

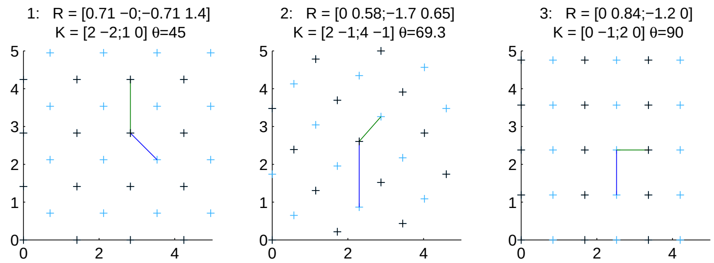

Figure 2: Three non-equivalent 2D lattices obtained for a design with dilation matrices having $|\det \mathbf{K}| = 2$. The lattice on the left is the well known quincunx sampling with a $\theta = 45°$ rotation. The other two are new schemes with different rotation angles. The black markers show the sample positions that are retained after sub-sampling by $\mathbf{K}$.

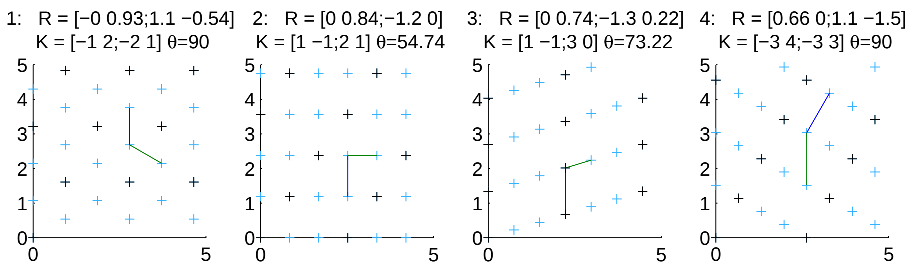

Figure 3: Three non-equivalent 2D lattices obtained for a design with dilation matrices having $|\det \mathbf{K}| = 3$. The lattice on the left is the well known hexagonal lattice at $\theta = 30°$ rotation. The other three are new schemes with different rotation angles.

$q(\mathbf{A}) = \{\mathbf{k}^T\mathbf{A}\mathbf{k} : \mathbf{k} \in \mathbb{Z}^n\}$ using the Gram matrices of the respective lattices. If the sorted lists $q(\mathbf{A}_1)$ and $q(\mathbf{A}_2)$ disagree in any element, $\mathbf{R}_1$ and $\mathbf{R}_2$ are not equivalent ([5], p. 60). It is possible to restrict the set of indices $\mathbf{k} \in \mathbb{Z}^n$ to the Voronoi relevant neighbours [1]. Further, since these neighbours determine the hyperplanes bounding the Voronoi polytope of the lattice, they can also be used for a sufficient test for equivalence.

3. Constructions for different dimensions and sub-sampling ratios

For the 2D case we have created lattices permitting a reduction rate 2 in Figure 2 and rate 3 in Figure 3. In both cases, familiar examples arise in the quincunx and the hex lattice for the respective ratios.

A search of 3D lattices enjoying the self-similar sub-sampling property with rate 2 dilations resulted in 53 non-equivalent cases. These lattices were compared in terms of their dimensionless second order moments, corresponding to the expected squared vector quantization error ([2], p. 451). When performing the continuous optimization mentioned at the end of Section 2.1, all of these cases converged to the same optimum lattice shown in Figure 4. The dimensionless second order moment for the Voronoi Cell of this lattice is $G = 0.081904$. For comparison, the Cartesian cube has $G_{cc} = 0.0833$ and the truncated octahedron of the BCC lattice has $G_{bcc} = 0.0785$.

4. Discussion and potential applications

The current formation of candidate matrices $\mathbf{K}$ based on similarity transforms of one valid example is not guaranteed to produce all possible solutions. For 2D and 3D we also employed an exhaustive search over a range of integer matrices with values in $[-3, 3]$ resulting in the same number of non-equivalent 2D cases as the construction via $\mathbf{K}_T$. However, for dimensionality $n > 3$ the exhaustive search had to be replaced by a random sampling of integer matrices ultimately rendering the method infeasible for $n > 5$. In that light the current construction via scaled eigenvectors of the companion matrix is a significant improvement as it allows to produce a large number of non-equivalent lattices for any dimensionality.

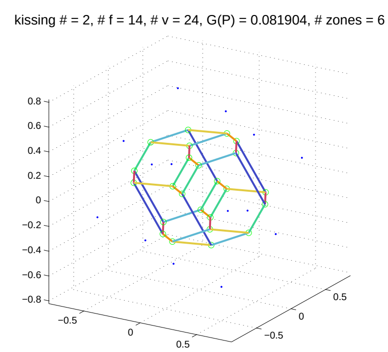

Figure 4: The best 3D lattice obtained for a design with dilation matrices having $\det \mathbf{K} = 2$.

Our sub-sampling schemes may have applications for multi-dimensional wavelet transforms [7]. Another direction for possible investigation is the construction of sparse grids that are employed in the context of high-dimensional integration and approximation adapting to smoothness conditions of the underlying function space [3].

Appendix: Characteristic polynomial of a rotation matrix in $\mathbb{R}^n$

The similarity relationship between $\mathbf{K}$ and $\mathbf{Q}$ in Equation 2 implies that they share the same characteristic polynomial $d(\lambda) = \det(\mathbf{K} - \lambda\mathbf{I}) = \det(\mathbf{Q} - \lambda\mathbf{I})$ leading to an agreement in eigenvalues $d(\lambda_k) = 0$ and determinant $d(0)$ ([6], p. 184). Further, since $\mathbf{K}$ is an integer matrix the polynomial $d(\lambda) \in \mathbb{Z}[\lambda]$ has integer coefficients c_k.
In order to find integer matrices $\mathbf{K}$ with the eigenvalues of a scaled rotation matrix, it will be important to distinguish the two different forms of the diagonal matrix $\mathbf{\Delta}$ in Equation 5 and 4 for the case $n =$ *even*

$$\mathbf{\Delta} = \text{diag}[e^{j\theta_1}\ e^{-j\theta_1}\ \ldots e^{j\theta_{n/2}}\ e^{-j\theta_{n/2}}]$$

and the case $n =$ *odd*

$$\mathbf{\Delta} = \text{diag}[1\ e^{j\theta_1}\ e^{-j\theta_1} \ldots e^{j\theta_{(n-1)/2}}\ e^{-j\theta_{(n-1)/2}}]$$

with analogue block-wise constructions for $\mathbf{J}_n$.
For dimensionality $n = even$ the characteristic polynomial fulfills

$$\begin{aligned} d(\lambda) &= \prod_{k=1}^{n/2} (\alpha e^{j\theta_k} - \lambda)(\alpha e^{-j\theta_k} - \lambda) \\ &= \prod_{k=1}^{n/2} (\alpha^2 - 2\lambda\alpha\cos\theta_k + \lambda^2) \\ &= \prod_{k=1}^{n/2} \left[(\frac{\alpha^4}{\lambda^2} - 2\frac{\alpha^3}{\lambda}\cos\theta_k + \alpha^2)\frac{\lambda^2}{\alpha^2} \right] \\ &= d\left(\frac{\alpha^2}{\lambda}\right)\left(\frac{\lambda}{\alpha}\right)^n \end{aligned} \tag{9}$$

Thus, if

$$\begin{aligned} d(\lambda) &= \sum_{k=0}^{n} c_k \lambda^k \\ &= \sum_{k=0}^{n} c_k \left(\frac{\alpha^2}{\lambda}\right)^k \left(\frac{\lambda}{\alpha}\right)^n \\ &= \sum_{k=0}^{n} c_{n-k}\alpha^{n-2k}\lambda^k \\ &\Leftrightarrow c_k = \alpha^{n-2k}c_{n-k} = \delta^{1-\frac{2k}{n}}c_{n-k}. \end{aligned} \tag{10}$$

If $c_k \neq 0$ and $c_k, \delta \in \mathbb{Z}$ then $\delta^{1-\frac{2k}{n}} \in \mathbb{Q}$. This is impossible for $0 < 2k < n$, assuming small values of δ, such as $2, 3$ or any simple product of primes. This implies that $c_k = c_{n-k} = 0$ for $k = 1, 2, \ldots \frac{n}{2} - 1$. For $k = \frac{n}{2}$ the c_k can be non-zero leading to

$$d(\lambda) = \lambda^n + C\lambda^{\frac{n}{2}} + \alpha^n \tag{11}$$

with the requirement that $C^2 < 4\alpha^n$ so that the complex eigenvalues $d(\lambda_k) = 0$ are evenly distributed on the complex circle of radius $|\lambda_k| = \alpha$.
For dimensionality $n = odd$ the polynomial fulfills

$$\begin{aligned} d(\lambda) &= (\alpha - \lambda) \prod_{k=1}^{(n-1)/2} (\alpha e^{j\theta_k} - \lambda)(\alpha e^{-j\theta_k} - \lambda) \\ \Rightarrow d(\lambda) &= -\left(\frac{\lambda}{\alpha}\right)^n d\left(\frac{\alpha^2}{\lambda}\right) \end{aligned} \tag{12}$$

Thus, if

$$\begin{aligned} d(\lambda) &= \sum_{k=0}^{n} c_k \lambda^k \\ &= -\sum_{k=0}^{n} c_k \left(\frac{\alpha^2}{\lambda}\right)^k \left(\frac{\lambda}{\alpha}\right)^n \\ &= -\sum_{k=0}^{n} c_{n-k}\alpha^{n-2k}\lambda^k \\ &\Leftrightarrow c_k = -\alpha^{n-2k}c_{n-k} = -\delta^{1-\frac{2k}{n}}c_{n-k}. \end{aligned} \tag{13}$$

By the same reasoning as for the even case, $c_k = 0$ for all $k = 1, 2, \ldots \frac{n-1}{2}$ resulting in only one possible characteristic polynomial

$$d(\lambda) = \lambda^n - \alpha^n. \tag{14}$$

To refer to the above procedure we will invoke a function compoly(n, α, C) that returns a companion matrix with a characteristic polynomial as in Equation 7 or 14.

References:

[1] E. Agrell, T. Eriksson, A. Vardy, and K. Zeger. Closest point search in lattices. *Information Theory, IEEE Transactions on*, 48(8):2201–2214, August 2002.

[2] J.H. Conway and N.J.A. Sloane. *Sphere Packings, Lattices and Groups. – 3rd ed.* Springer, 1999.

[3] M. Griebel. Sparse grids and related approximation schemes for higher dimensional problems. In L. Pardo, A. Pinkus, E. Suli, and M.J. Todd, editors, *Foundations of Computational Mathematics (FoCM05), Santander*, pages 106–161. Cambridge University Press, 2006.

[4] Y.M. Lu, M.N. Do, and R.S. Laugesen. A Computable Fourier Condition Generating Alias-Free Sampling Lattices. *IEEE Transactions on Signal Processing, to appear*, 2008.

[5] M. Newman. *Integral Matrices*. Academic Press, 1972. See `http://www.dleex.com/read/?3907` for a digital copy.

[6] L.N. Trefethen and D. Bau III. *Numerical Linear Algebra*. SIAM, 1997.

[7] D. Van De Ville, T. Blu, and M. Unser. Isotropic polyharmonic B-Splines: Scaling functions and wavelets. *IEEE Transactions on Image Processing*, 14(11):1798–1813, November 2005.

[8] D. Van De Ville, T. Blu, and M. Unser. On the multidimensional extension of the quincunx subsampling matrix. *IEEE Signal Processing Letters*, 12(2):112–115, February 2005.

[9] E. Viterbo and E. Biglieri. Computing the Voronoi cell of a lattice: The diamond-cutting algorithm. *Information Theory, IEEE Trans. on*, 42(1):161–171, 1996.