HIGH-QUALITY CAUSAL INTERPOLATION FOR ONLINE UNIDIMENSIONAL SIGNAL PROCESSING

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ABSTRACT

We present a procedure for designing interpolation kernels that are adapted to time signals; i.e., they are causal, even though they do not have a finite support. The considered kernels are obtained by digital IIR filtering of a finite support function that has maximum approximation order. We show how to build these kernel starting from the all-pole digital filter and we give some practical design examples.

1. MOTIVATION

Interpolation, whose historical roots can be traced back to the first known literate societies [1], is a central tool in engineering sciences. This is especially true in digital signal processing applications where, implicitly or explicitly, an interpolation step is frequently required. Unfortunately, the "ideal" interpolation kernel, the sinc-function, which is used in Shannon's interpolation formula (T is some sampling step)

$$f(t) = \sum_{n\in\mathbb{Z}} f_n \operatorname{sinc}\left(\frac{t}{T} - n\right), \quad (1)$$

has three weak points which rule it out for real-world systems. More specifically, it does not satisfy any of the following properties:

i. Finite cost implementation — Instead, the sinc has infinite support with no finite (e.g., recursive) implementation;

ii. Causality — Instead, sinc-interpolation requires all past and all future samples;

iii. Absolute summability — Which not the case of the sinc function because $\sum_{n\in\mathbb{Z}} |\operatorname{sinc}(t/T - n)| = +\infty$, implying excessive sensitivity to additive noise.

As a remedy, practitionneers have tried to replace sinc in (1) by less ideal — but carefully designed — kernels that would fulfill these three requirements. In fact, it has become standard to look for kernels $\varphi_{\text{int}}(t)$ that have finite support and satisfy the interpolation condition $\varphi_{\text{int}}(n) = \delta_n$ [2, 3, 4].

Generalized interpolation — In previous publications, we have shown that the standard approach yields significantly suboptimal kernels [5, 6]. We exhibited infinite support kernels satisfying properties *i.* and *iii.*, that are competitive with regards to computational cost and have optimal approximation quality (MOMS) [7]. Our approach consists in building the interpolating kernel $\varphi_{\text{int}}(t)$ from a *finite support* function $\varphi(t)$ according to the expression

$$\varphi_{\text{int}}(t) = \sum_{k\in\mathbb{Z}} h_k \varphi(t-k)$$
$$\text{where } H(e^{j\omega}) = \Big(\sum_{n\in\mathbb{Z}} \varphi(n) e^{-jn\omega}\Big)^{-1} \quad (2)$$

Here, $H(z) = (\sum_{n\in\mathbb{Z}} \varphi(n) z^{-n})^{-1}$ is the z-transform of the digital filter $\{h_k\}_{k\in\mathbb{Z}}$. It is an all-pole fractional transfer function which can be implemented *exactly* by a recursive algorithm (see [6].

It can easily be checked that $\varphi_{\text{int}}(n) = \delta_n$ even though $\varphi(t)$ does not satisfy the interpolation condition.

Causal interpolation — When dealing with images, it is pointless to enforce the causality of the interpolant $\varphi_{\text{int}}(t)$. On the contrary, when time signals are considered, this property is essential. It is precisely the purpose of the present paper to adapt generalized interpolation so that it satisfies property *ii.* We will, in particular, show how to build kernels $\varphi(t)$ that have minimum support for a given approximation order and whose prefilter $H(z)$ is causal. We will finally conclude with a few good candidates that have both a fast implementation, and good approximation qualities.

We believe that the new interpolation kernels we propose here can be of great use in practical applications such as audio sound resampling (e.g., 48 kHz studio recording down to 44.1 kHz CD-quality, or 192 kHz DVD sampling down to 44.1 kHz CD-quality).

2. MOMS

The order of approximation denoted by L gives the rate at which the interpolation error decreases as the sampling step T decreases:

$$\Big\| f(t) - \sum_{n\in\mathbb{Z}} f_n \varphi_{\text{int}}\Big(\frac{t}{T} - n\Big)\Big\|_{\mathbf{L}^2} \underset{T\to 0}{\propto} T^L \quad (3)$$

where f is some function with square-integrable L^{th} derivative. In a series of image rotation experiments also available on our web page[1], we have been able to show that the approximation order is an important quantity for comparing the quality of different interpolation kernels.

Mathematically, $\varphi_{\text{int}}(t)$ as defined in (2) generates an L^{th}-order interpolation if and only if the function $\varphi(t)$ satisfies the so-called Strang-Fix conditions:

$$\hat{\varphi}(0) = 1,$$
$$\hat{\varphi}(\omega + 2n\pi) = O(\omega^L) \quad \text{for all } n \in \mathbb{Z}\setminus\{0\}.$$

We use here the notation $g(x) = O(x^L)$ as a shorthand meaning that the Taylor development of $g(x)$ in the neighborhood of $x = 0$ is of the form: $g(x) = 0 + \ldots + 0\cdot x^{L-1} + a_L x^L + \ldots$

Not so surprisingly, a function $\varphi(t)$ of given support cannot have an arbitrarily large approximation order [8]: if the

[1] http://bigwww.epfl.ch/demo/jrotation/index.html

size of its support is S, then its maximal approximation order is $L = \lfloor S \rfloor \le S$. In addition, the functions that saturate the inequality, i.e., such that $L = S$, are piecewise-polynomial and can be expressed as a sum of the B-spline of degree $L-1$ and its first $L-1$ derivatives:

$$\varphi(t) = \beta^{L-1}(t-\tau) + \sum_{k=1}^{L-1} p_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \beta^{L-1}(t-\tau) \tag{4}$$

where τ is an arbitrary shift and p_k are arbitrary coefficients — This equation characterizes the MOMS (Maximum Order Minimum Support) function class [7]. We recall that the B-spline of degree d is a positive piecewise-polynomial function whose support is $[0, d+1]$ and whose Fourier transform is given by

$$\hat{\beta}^d(\omega) = \Big(\frac{1-e^{-j\omega}}{j\omega}\Big)^{d+1}.$$

The parameters of a MOMS can be designed so that it satisfies the interpolation condition (I-MOMS or piecewise Lagrange kernel), or so that the proportionality constant in (3) — also known as the asymptotic constant — is minimal (O-MOMS).

3. MOMS WITH CAUSAL PREFILTER

Usually, the all-pole filter $H(z) = (\sum_{n\in\mathbb{Z}} \varphi(n) z^{-n})^{-1}$ cannot have a causal recursive implementation. Although this is not an issue in image interpolation, this may be detrimental in the case of online unidimensional signal processing. This is why we want to characterize the subclass of MOMS that have a causal prefilter $H(z)$.

Our strategy is the following: Instead of finding which constraint the parameters of the MOMS (τ and p_k) have to satisfy in order for $H(z)$ to be causal, we will instead assume that we are given τ and the poles of $H(z)$ — inside the unit circle — and show that this uniquely determines the coefficients p_k. To this end, we define

$$A(z) = \sum_{k=0}^{L-1} a_k z^{-k}$$

as the polynomial whose zeros are the poles of $H(z)$ and we assume that the coefficients a_k are normalized by $A(1) = 1$. Notice that we also have $\sum_k \varphi(k) = 1$ because $\varphi(t)$ satisfies the Strang-Fix condition of order $L \ge 1$. It is thus easy to verify that

$$\sum_{n\in\mathbb{Z}} \varphi(n) z^{-n} = z^{-n_0} A(z)$$

where n_0 is some integer delay. With no lack of generality, we may assume that τ has been chosen in such a way that $n_0 = 0$ but then, this implies that τ is not completely arbitrary anymore because it has to satisfy

$$\deg(A) - L < \tau \le 0. \tag{5}$$

Theorem 1 *Assume that the shift τ and the poles of the generalized interpolation prefilter are provided (under the form of a normalized polynomial $A(z)$), and that they satisfy the constraint* (5).

Then, the unique MOMS function $\varphi(t)$ such that

$$\sum_{n\in\mathbb{Z}} \varphi(n) z^{-n} = A(z)$$

is given by (4) *where the parameters p_k are the coefficients of the Taylor development of the function*

$$\frac{e^{j\tau\omega} A(e^{j\omega})}{\hat{\beta}^{L-1}(\omega)} = 1 + p_1 j\omega + p_2 (j\omega)^2 + \dots p_{L-1} (j\omega)^{L-1} + O(\omega^L) \tag{6}$$

Proof: Using Poisson's summation formula, we have that

$$\sum_{n\in\mathbb{Z}} \hat{\varphi}(\omega + 2n\pi) = \sum_{n\in\mathbb{Z}} \varphi(n) e^{-jn\omega} = A(e^{j\omega}).$$

Applying the Strang-Fix conditions $\hat{\varphi}(\omega + 2n\pi) = O(\omega)$ for $n \ne 0$ to this expression shows that $\hat{\varphi}(\omega) = A(e^{j\omega}) + O(\omega^L)$. We then take the Fourier transform of (4) which provides $\hat{\varphi}(\omega) = e^{-j\tau\omega} P(j\omega) \hat{\beta}^{L-1}(\omega)$, where we have denoted $P(x) = 1 + p_1 x + \dots p_{L-1} x^{L-1}$. After dividing by $e^{-j\tau\omega}\hat{\beta}^{L-1}(\omega)$ we get (6).

Conversely, if (6) is satisfied, then we build the function $\varphi(t)$ according to (4). We only have to verify that $\sum_n \varphi(n) z^{-n} = A(z)$. Because of (6), we are ensured that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \varphi(n) e^{-jn\omega} &= e^{-j\lceil\tau\rceil\omega} \sum_{n\in\mathbb{Z}} \varphi(n + \lceil\tau\rceil) e^{-jn\omega} \\ &= A(e^{j\omega}) + O(\omega^L) \\ &= A(e^{j\omega}) + O((1-e^{-j\omega})^L)\end{aligned}$$

which provides

$$\sum_{n\in\mathbb{Z}} \varphi(n + \lceil\tau\rceil) e^{-jn\omega} - e^{j\lceil\tau\rceil\omega} A(e^{j\omega}) = O((1-e^{-j\omega})^L).$$

The first term on the left-hand side is a polynomial of degree $(L-1)$, and so is the second term because τ satisfies (5). As a result, the lhs is a polynomial of degree $(L-1)$ in $e^{-j\omega}$ which should be divisible by a polynomial of degree L. This is only possible if the lhs vanishes, i.e.,

$$\sum_{n\in\mathbb{Z}} \varphi(n + \lceil\tau\rceil) e^{-jn\omega} = e^{j\lceil\tau\rceil\omega} A(e^{j\omega})$$

which finally implies that $A(e^{j\omega}) = \sum_n \varphi(n) e^{-jn\omega}$. □

Note that the function specified by this theorem is usually not continuous because the $(L-1)^{\text{th}}$ derivative of a spline of degree $(L-1)$ is not continuous at the integers. This means that, if we are looking for more regular kernels, we will need to look for solutions of (6) which are such that: $p_{L-1} = 0$ (continuity), $p_{L-1} = 0$ and $p_{L-2} = 0$ (continuous first differentiation), and so on.

4. SOME DESIGN EXAMPLES

Theorem 1 shows how to choose the MOMS function $\varphi(t)$ so that the the interpolation prefilter $H(z)$ is given by $A(z)^{-1}$. We can now make a choice for $A(z)$. More specifically, we are interested in 1-pole interpolation prefilters because they have the fastest implementation. Since the root of $A(z)$ has to be inside the unit circle, we are thus left with the expression:

$$A(z) = \alpha + (1-\alpha) z^{-1} \quad \text{with } \alpha > 1/2.$$

According to (5), τ has to be chosen in $]1-L, 0]$ — we assume $\alpha \ne 1$ here.

We will consider three interesting values of L corresponding to a piecewise-polynomial kernel of degree $L-1$.

4.1 Approximation order $L = 2$.

The corresponding MOMS is a piecewise-linear function. Applying Theorem 1, we get

$$p_1 = \alpha + \tau.$$

For the kernel to be continuous, we thus need that $\tau = -\alpha$, that is to say that we have a free parameter left $\alpha \in]1/2, 1[$, yielding

$$\varphi(t) = \beta^1(t+\alpha).$$

This degree of freedom can be put to benefit for optimizing the quality of interpolation. More specifically, when we minimize the proportionality constant in (3), we get $\alpha \approx 0.79$ (see [9]). We have plotted the corresponding interpolating function $\varphi_{\text{int}}(t)$ in Fig. 1 and its spectrum in Fig. 4.

4.2 Approximation order $L = 3$.

The corresponding MOMS is a piecewise-quadratic function. Applying Theorem 1, we get:

$$\begin{aligned} p_1 &= \tfrac{1}{2} + \alpha + \tau \\ p_2 &= \tfrac{1}{2}(1+\tau)(2\alpha+\tau). \end{aligned}$$

In order to have a continuous kernel, we have two options:

- either $\tau = -2\alpha$ in which case $p_1 = -(\alpha - 1/2)$ and $\alpha \in]1/2, 1[$, yielding

$$\varphi(t) = \beta^2(t+2\alpha) - (\alpha - 1/2)\frac{\mathrm{d}}{\mathrm{d}t}\beta^2(t+2\alpha);$$

- or $\tau = -1$ in which case $p_1 = \alpha - 1/2$ and $\alpha \in]1/2, +\infty[$, yielding

$$\varphi(t) = \beta^2(t+1) + (\alpha - 1/2)\frac{\mathrm{d}}{\mathrm{d}t}\beta^2(t+1).$$

This provides two families with one free parameter. We have plotted the interpolating function $\varphi_{\text{int}}(t)$ corresponding to the first option with $\alpha = 3/4$ in Fig. 1 and its spectrum in Fig. 4.

4.3 Approximation order $L = 4$.

The corresponding MOMS is a piecewise-cubic function. Applying Theorem 1, we get:

$$\begin{aligned} p_1 &= 1 + \alpha + \tau \\ p_2 &= 1/3 + \tau + 3\alpha/2 + \tau^2/2 + \alpha\tau \\ p_3 &= (1+\tau)(2+\tau)(3\alpha+\tau)/6. \end{aligned}$$

Interestingly, it is possible here to have a continuously differentiable kernel which yields a unique solution ($\tau = -2$ and $\alpha = 2/3$):

$$\varphi(t) = \beta^3(t+2) - \frac{1}{3}\frac{\mathrm{d}}{\mathrm{d}t}\beta^3(t+2).$$

Even more interesting is that the generalized interpolation prefilter is $\left(2/3(1+z^{-1}/2)\right)^{-1}$ which can be implemented using a recursion that requires only one division by 2 and one addition per sample. This could be particularly useful for fixed arithmetic implementation.

We have plotted the corresponding interpolating function $\varphi_{\text{int}}(t)$ in Fig. 3 and its spectrum in Fig. 4.

Preliminary image rotation tests have shown that this cubic kernel behaves even better than cubic spline interpolation.

5. CONCLUSION

We have presented a general technique for designing infinite support causal interpolators based on the shortest functions that have a given approximation order L; these turn out to be piecewise-polynomial of degree $L-1$. We have, in particular, considered the case of the simplest interpolation prefilter and have analyzed the regular solutions obtained when the polynomial degree of the interpolator is 1, 2 and 3. We believe that, due to their low implementation cost (see [6]) and their good approximation properties, these interpolants are likely to be useful especially for the interpolation of time signals.

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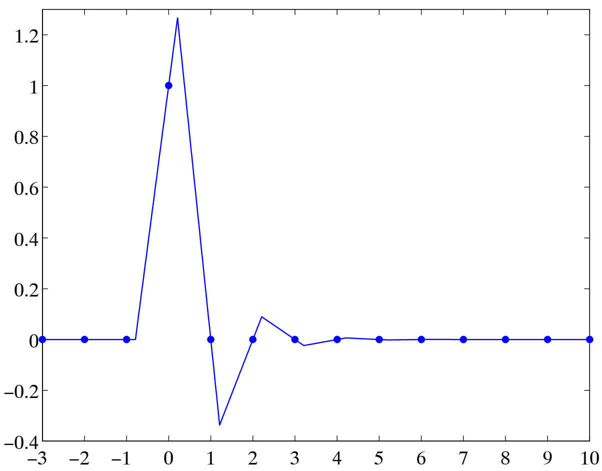

Figure 1: Interpolating linear function $\varphi_{\text{int}}(t)$ in the case $L = 2$ for the optimal value $\alpha = 0.79$ [7].

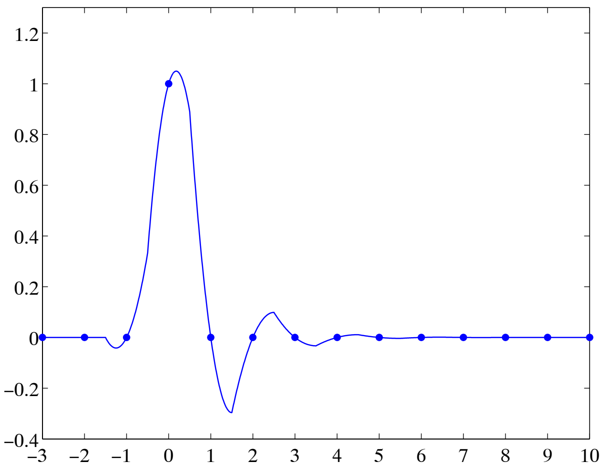

Figure 2: Interpolating quadratic function $\varphi_{\text{int}}(t)$ in the case $L = 3$ for the values $\alpha = 0.79$ and $\tau = -2\alpha$.

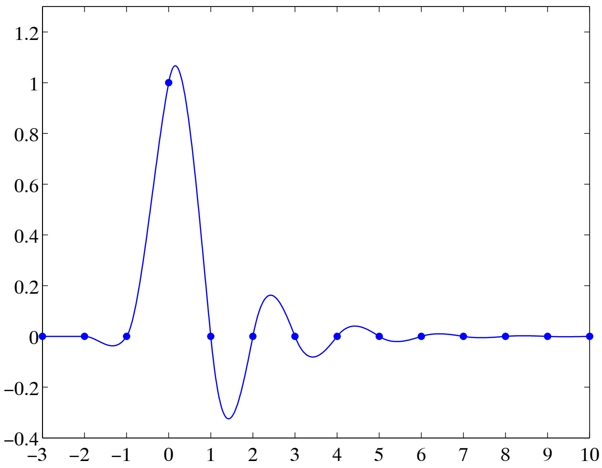

Figure 3: Continuously differentiable interpolating cubic function $\varphi_{\text{int}}(t)$ in the case $L = 4$.

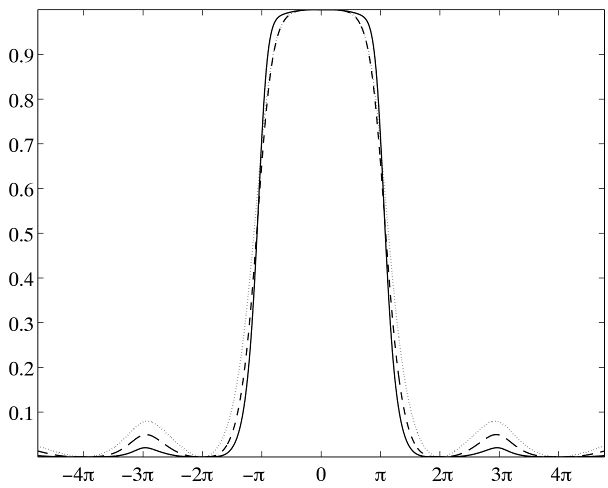

Figure 4: Frequency spectrum of the three functions plotted in Figs 1–3. Dotted line: corresponding to Fig. 1; Dashed line: corresponding to Fig. 2; Plain line: corresponding to Fig. 3.