HOW A SIMPLE SHIFT CAN SIGNIFICANTLY IMPROVE THE PERFORMANCE OF LINEAR INTERPOLATION

Thierry Blu, Philippe Thévenaz and Michael Unser

Biomedical Imaging Group FSTI/IOA
Swiss Federal Institute of Technology Lausanne (EPFL)
CH-1015 Lausanne EPFL, Switzerland.

e-mail: {thierry.blu, philippe.thevenaz, michael.unser}@epfl.ch

ABSTRACT

We present a simple, original method to improve piecewise linear interpolation with uniform knots: We shift the sampling knots by a fixed amount, while enforcing the interpolation property.

Thanks to a theoretical analysis, we determine the optimal shift that maximizes the quality of our shifted linear interpolation. Surprisingly enough, this optimal value is nonzero and it is close to $1/5$.

We confirm our theoretical findings by performing a cumulative rotation experiment, which shows a significant increase of the quality of the shifted method with respect to the standard one. Most interesting is the fact that we get a quality similar to that of high-quality cubic convolution at the computational cost of linear interpolation.

1 INTRODUCTION

Interpolation is essential in many digital image- and signal-processing applications [1, 2]. Standard piecewise linear interpolation which dates back to Babylonians, is by far the most popular solution for many applications (e.g., vision, digital photography, graphics data processing, postscript optimization for printers, image calibration and registration, textures, re-sampling) because it is reasonably fast (and does not suffer from the obvious blocking artifacts of nearest-neighbour interpolation).

However, when quality is an important concern, methods based on higher-degree interpolation kernels have been developped: Keys' cubic convolution method [3] has become as a standard in the field, even though recent studies have shown that, for the same computational cost, cubic spline-based kernels provide a substantial gain in quality [4, 5].

The method that we are proposing here amounts to shifting the standard linear interpolation kernel. In particular, we show that there exists an optimal, non-trivial shift value (close to 1/5) for linear interpolation, for which our new shifted interpolation substantially improves the standard—nonshifted—method. To evaluate this quality, we rely on theoretical tools developped in [6], and perform experiments.

Our method provides a quality at least equal and even higher than cubic convolution, but at a much lower computational cost—that of linear interpolation. In other words, one can get top quality at the cost of a low-end method.

2 INTERPOLATION

Given a sequence of samples $f_n = f(nT)$ originating from the uniform sampling of a function $f(x)$ with step T, standard linear interpolation builds a function $f_T(x)$ through the process: for $x \in [(n-1)T, nT]$, $f_T(x) = a_n x + b_n$, where a_n and b_n are chosen such that $f_T((n-1)T) = f_{n-1}$ and $f_T(nT) = f_n$. This can be shown to be equivalent to

$$f_T(x) = \sum_{n\in\mathbb{Z}} f_n \Lambda\Big(\frac{x}{T} - n\Big), \tag{1}$$

where $\Lambda(x)$ is the "hat" function: $\Lambda(x) = 1-|x|$ for $|x| \le 1$ and $\Lambda(x) = 0$ for $|x| > 0$.

In a more general way, uniform interpolation is the process of building a function $f(x)$ through the formula

$$f_T(x) = \sum_{n\in\mathbb{Z}} c_n \varphi\Big(\frac{x}{T} - n\Big), \tag{2}$$

where the coefficients c_n are chosen so as to satisfy the interpolation condition $f_T(nT) = f_n$. Here, $\varphi(x)$ might be any function with $\int \varphi(x)\,\mathrm{d}x = 1$. As can readily be seen, the interpolation condition is equivalent to the following filtering relation

$$f_k = \sum_{n\in\mathbb{Z}} \varphi(k-n) c_n\,.$$

Thus, the c_n's can be obtained by convolving the f_n's with the filter whose z-transform is $\frac{1}{\sum_n \varphi(n) z^{-n}}$.

For $f_T(x)$ to be a good approximation of $f(x)$, the quality needs to improve as T gets smaller; the rate of this improvement is called approximation order. Usually, we assume that $f \in \mathbf{W}_2^{1/2+\varepsilon}$ (with $\varepsilon > 0$), so as to ensure that f and f_T belong to $\mathbf{L}^2$ [6]. A natural measure for the distance between $f_T(x)$ and $f(x)$ is then $\|f - f_T\|_{\mathbf{L}^2}$. Obviously, this approximation error is bounded from below by $\|f - \mathscr{P}_T f\|_{\mathbf{L}^2}$, where $\mathscr{P}_T f$ is the orthogonal projection of f onto the function space made of linear combinations of $\varphi(x/T - n)$, $n \in \mathbb{Z}$.

2.1 Asymptotic Constant

When the sampling step tends to 0, we want that $f_T(x) \to f(x)$. It is known that $\|f - \mathscr{P}_T f\|_{\mathbf{L}^2}$ tends to zero as T^L if and only if $\varphi(x)$ satisfies the Strang-Fix conditions [7, 8]: $\hat{\varphi}^{(l)}(2n\pi) = 0$ for $n \in \mathbb{Z} \setminus \{0\}$ and $l = 0 \ldots L-1$. This equivalence still holds for $\|f - f_T\|_{\mathbf{L}^2}$. The integer L is called the approximation order of $\varphi(x)$; for instance, the approximation order of $\Lambda(x)$ is 2.

Furthermore, one of us showed that

$$\|f - \mathscr{P}_T f\|_{\mathbf{L}^2} \approx C_\varphi^- \, \|f^{(L)}\|_{\mathbf{L}^2} \, T^L$$

as $T \to 0$, where the constant C_φ^- is given by the following expression [9]:

$$C_\varphi^- = \frac{1}{L!}\sqrt{\sum_{n \neq 0} |\hat{\varphi}^{(L)}(2n\pi)|^2}\,. \tag{3}$$

Similarly, $T^{-L}\|f - f_T\|_{\mathbf{L}^2} \to C_\varphi^{\text{int}} \|f^{(L)}\|_{\mathbf{L}^2}$ as $T \to 0$ with [10, 6, 4]

$$C_\varphi^{\text{int}} = \sqrt{\frac{1}{L!^2}\left|\sum_{n \neq 0} \hat{\varphi}^{(L)}(2n\pi)\right|^2 + (C_\varphi^-)^2}\,, \tag{4}$$

where it clearly appears that the quantity $\sum_{n \neq 0} \hat{\varphi}^{(L)}(2n\pi)$ tells apart interpolation from orthogonal projection.

3 SHIFTED LINEAR INTERPOLATION

Instead of building $f_T(x)$ using line segments between $(n-1)T$ and nT as in Section 2, we draw these line segments between $(n-1+\tau)T$ and $(n+\tau)T$ for some $\tau \in [0, 1/2[$, as exemplified in Fig. 1; i.e., we consider the following process: for $x \in [(n-1+\tau)T, (n+\tau)T]$, $f_T(x) = a_n x + b_n$ where a_n and b_n are chosen such that $f_T(nT) = f_n$ and $f_T(x)$ is continuous at $x = (n-1+\tau)T$. This is equivalent to the following interpolation formula

$$f_T(x) = \sum_{n \in \mathbb{Z}} c_n \Lambda\left(\frac{x}{T} - n - \tau\right) \tag{5}$$

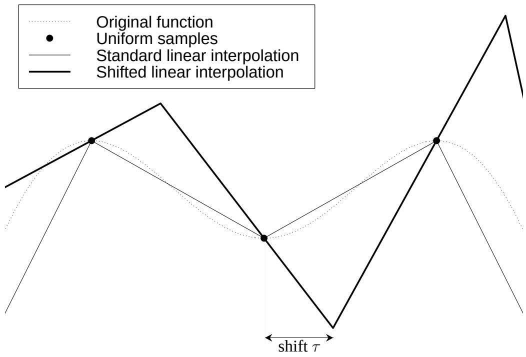

Fig. 1. "Shifted" versus "standard" linear interpolation; here, $\tau = 0.4$ which is far from the optimum (8).

$$\text{with} \quad c_n = -\frac{\tau}{1-\tau} c_{n-1} + \frac{1}{1-\tau} f_n. \tag{6}$$

That is to say, the function $\varphi(x)$ to be used in (2) is the shifted hat function, $\Lambda(x - \tau)$. Note that, because of the choice $\tau \in [0, 1/2[$, the filter that relates f_n to c_n is causal. However, unlike standard linear interpolation ($\tau = 0$), this filter is not FIR even though its impulse response decreases exponentially (see Fig. 2). As shown by (6), the implementation of this IIR 1-pole prefilter can be realized very efficiently and requires only 2 multiplications and one addition.

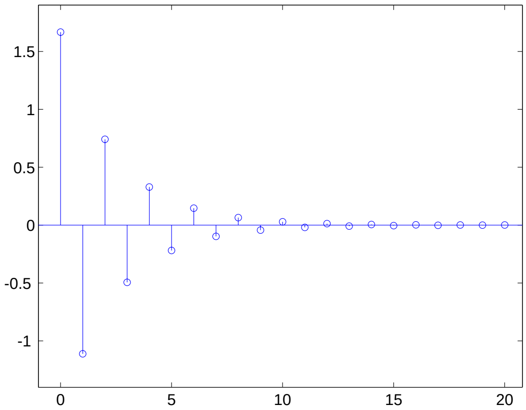

Fig. 2. Impulse response of the prefilter of the shifted linear interpolation of Fig. 1.

3.1 Optimal Shift

We have determined the asymptotic interpolation constant of the shifted linear scheme:

$$C_\tau^{\text{int}} = \sqrt{\frac{1}{4}\left(\tau^2 - \tau + \frac{1}{6}\right)^2 + \frac{1}{720}}. \tag{7}$$

Surprisingly, this expression is not minimized for the standard linear interpolation $\tau = 0$. Instead, choosing $\tau = \tau_{\text{opt}}$

with

$$\tau_{\text{opt}} = \frac{1}{2}\left(1 - \frac{\sqrt{3}}{3}\right) \approx 0.21 \tag{8}$$

we find not only that the interpolation constant is minimized, but also that the optimal $\mathbf{L}^2$ approximation constant is reached through the shifted interpolation method.

Using the definition of the asymptotic constants, we can predict that, asymptotically as the sampling step tends to 0, the gain of shifted over standard linear interpolation is approximately 8 dB.

Obviously, this performance should degrade as the frequency content of the function to interpolate gets richer, that is, when the energy at higher frequencies becomes more significant. In particular, when $f(x)$ is the step function (a limit case that does not belong to $\mathbf{L}^2$, but for which we can still test our interpolation method), the shifted linear interpolation gives rise to a Gibbs phenomenon—unlike the standard method (see Fig. 3).

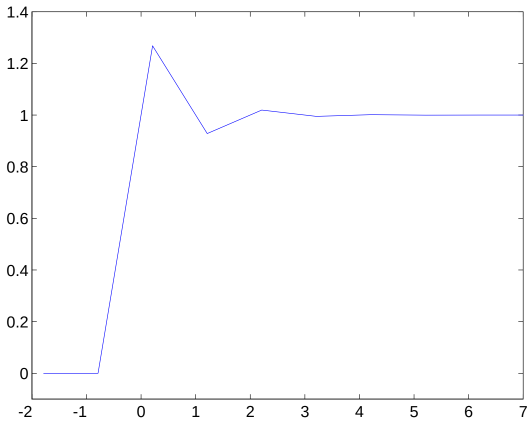

Fig. 3. Gibbs phenomenon caused by the—optimally—shifted linear interpolation of a unit step function.

4 SIMULATIONS AND PRACTICAL RESULTS

Although shifted linear interpolation seems counterintuitive, in particular because it dissymmetrizes a naturally symmetric method, it does in practice behave better than the non-shifted method. One of the reasons is that it tends to distribute more evenly the interpolation errors, as shown by the histograms in Fig. 4.

4.1 Rotation Experiments

In order to validate our theory, we devised a compounded rotation experiment of the ubiquitous Lena image. Let $f(x, y)$ denote this original image. We have access to its samples $\{f(k, l)\}_{k,l\in\mathbb{Z}}$ only. We first interpolate them—in a seperable fashion—to get $f_T(x, y)$ (here, $T = 1$); then, we rotate f_T by the angle $\theta = \frac{2\pi}{15}$, which provides $g(x, y) =$

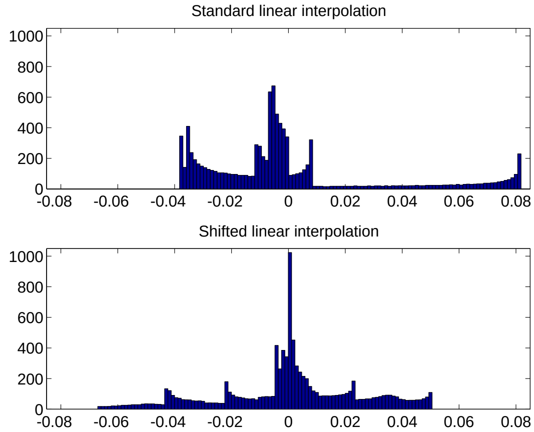

Fig. 4. Optimally-shifted versus standard linear interpolation: comparison of the error histograms—counts of the values of $(f(x) - f_T(x))$—resulting from the interpolation of the function of Fig. 1. Notice the strong dissymmetrical distribution of the errors in the standard method.

$f_T(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$; finally, we resample $g(x, y)$ on the original uniform grid, which gives the "rotated" image $\{g(k, l)\}_{k,l\in\mathbb{Z}}$. Iterating this procedure 15 times provides an image that has been rotated by $15 \times 24^\circ = 360^\circ$ degrees, and that can be readily compared to the original image. As is apparent from Fig. 5, the standard linear interpolation suffers from blurring, an effect that is avoided in the shifted method which provides much more details. More surprisingly, the shifted method appears to reach a quality that is pretty similar to that of the higher-order, more costly cubic interpolation [3], which is the reference high-quality method.

5 CONCLUSION

We have presented a simple, surprisingly powerful method for improving the performance of standard linear interpolation. For efficient implementation, we propose to precompute the model coefficients in a preprocessing step (simple recursive filtering), which amounts to replacing the initial data by a resampled version at the shifted knot location. With such a set-up, the method can be implemented directly via standard linear interpolation, so that we can readily take advantage of existing software or specialized hardware solutions.

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original image

standard linear interpolation
SNR=22 dB, CPU=5.1 s

τ_{opt}-shifted linear interpolation
SNR=28.2 dB, CPU=6.2 s

Keys' cubic interpolation
SNR=28.2 dB, CPU=9.4 s

Fig. 5. 15 successive rotations by 24° of Lena using standard, shifted linear, and Keys' interpolations: Notice the sharpness of the shifted method, as compared to the two others.

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