Approximation order of the LAP optical flow algorithm

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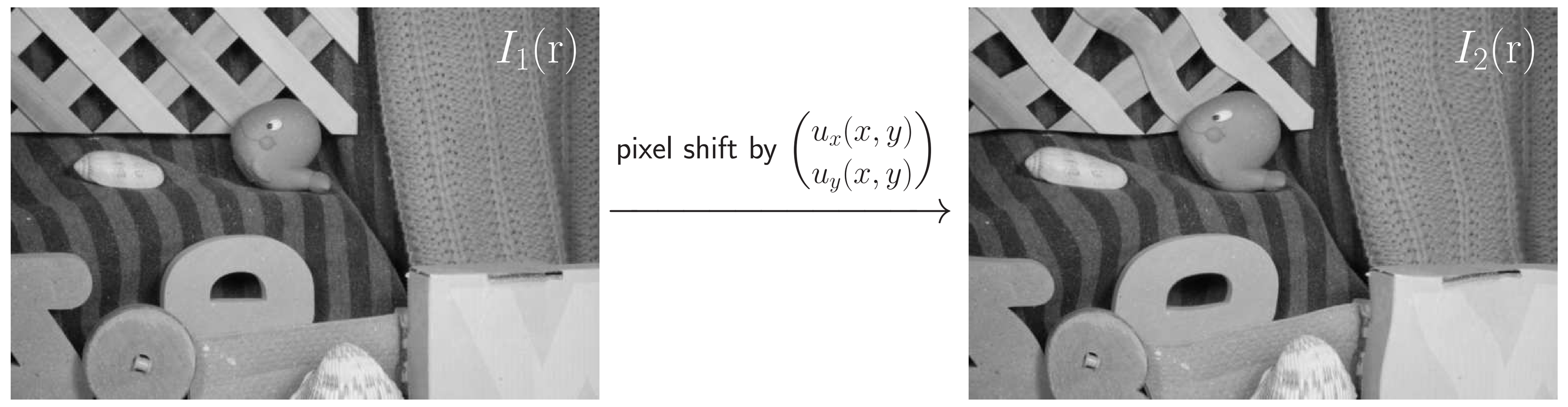
Summary

Estimating the displacements between two images is often addressed using a small displacement assumption, which leads to what is known as the optical flow equation. We study the quality of the underlying approximation for the recently developed Local All-Pass (LAP) optical flow algorithm, which is based on another approach—displacements result from filtering. While the simplest version of LAP computes only first-order differences, we show that the order of LAP approximation is quadratic, unlike standard optical flow equation based algorithms for which this approximation is only linear. More generally, the order of approximation of the LAP algorithm is twice larger than the differentiation order involved. The key step in the derivation is the use of Padé approximants.

Optical flow



Brightness constancy

The vector function $\mathrm{u}(\mathrm{r}) = \begin{pmatrix} u_x(x,y) \\ u_y(x,y) \end{pmatrix}$ satisfies $I_2(\mathrm{r}) = I_1(\mathrm{r} - \mathrm{u}(\mathrm{r}))$.

Standard algorithms (Lucas-Kanade, Horn-Schunck) are based on a linearization of the brightness constancy equation—the optical flow equation:

$$I_2(\mathrm{r}) = I_1(\mathrm{r}) - \mathrm{u}(\mathrm{r})^{\mathrm{t}} \nabla I_1(\mathrm{r}) + \mathrm{O}\big(\|\mathrm{u}(\mathrm{r})\|^2\big),$$

under a small $\|u(\mathrm{r})\|$ hypothesis; i.e., it is an approximation of order 1.

LAP optical flow estimation

The Local All-Pass (LAP) algorithm is based on the principle "shifting = filtering": when $\mathrm{u}(\mathrm{r}) = \mathrm{u}$ does not depend on r, we have

$$I_2(\mathrm{r}) = h(\mathrm{r}) * I_1(\mathrm{r})$$

where $h(\mathrm{r}) = \delta(\mathrm{r} - \mathrm{u})$ is, obviously, an **all-pass filter**. The second principle is that any all-pass filter can be expressed as $h(\mathrm{r}) = p(\mathrm{r}) * p^{-1}(-\mathrm{r})$, where $p(\mathrm{r})$ is arbitrary.

LAP optical flow equation

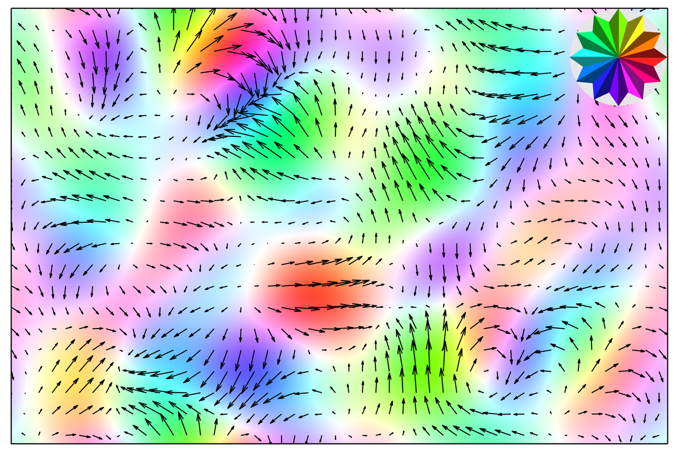
Brightness constancy is equivalent to a filtering equation $p(-\mathrm{r}) * I_2(\mathrm{r}) = p(\mathrm{r}) * I_1(\mathrm{r})$, where $p(\mathrm{r})$ is a space-varying filter.

The LAP algorithm essentially consists in **approximating** the spatially varying filter $p(\mathrm{r})$ and converting it into $\mathrm{u}(\mathrm{r})$. Method: express $p(\mathrm{r})$ locally as a linear combination of derivatives (up to order n) of Gaussian functions

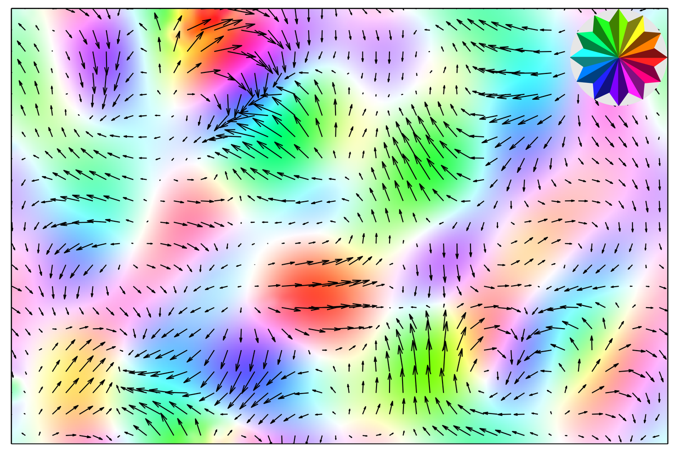
$$p(\mathrm{r}) = \sum_{l=0}^{n} \sum_{k=0}^{l} a_{k,l} \frac{\partial^l}{\partial x^k \partial y^{l-k}} \left\{ \exp\Big(-\frac{x^2+y^2}{2\sigma^2}\Big) \right\},$$

then solve for the unknown coefficients $a_{k,l}$ by minimizing the means-square LAP equation in a block around r, and finally convert to the local value of $\mathrm{u}(\mathrm{r})$.

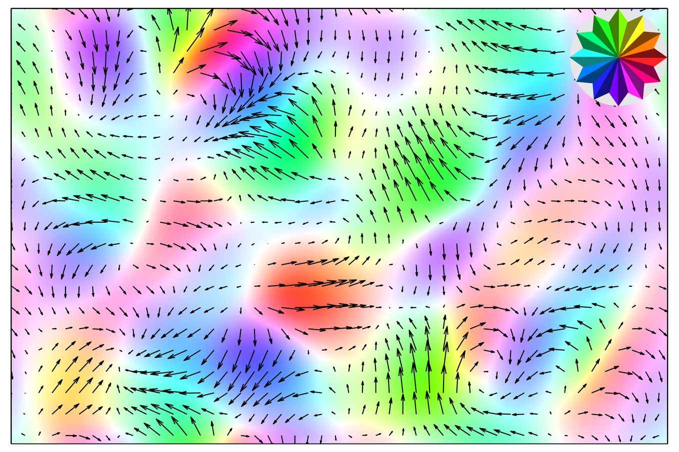
The LAP is very fast and accurate, outperforms the state of the art when brightness constancy is satisfied. Example of optical flow retrieved:



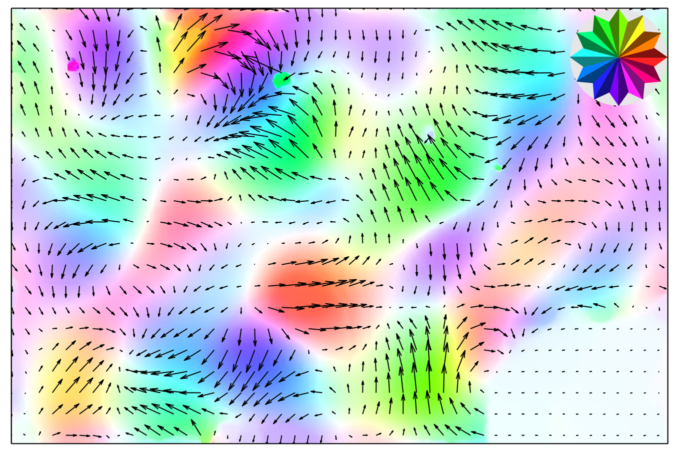
Ground-truth displacement field (15 pixels max)



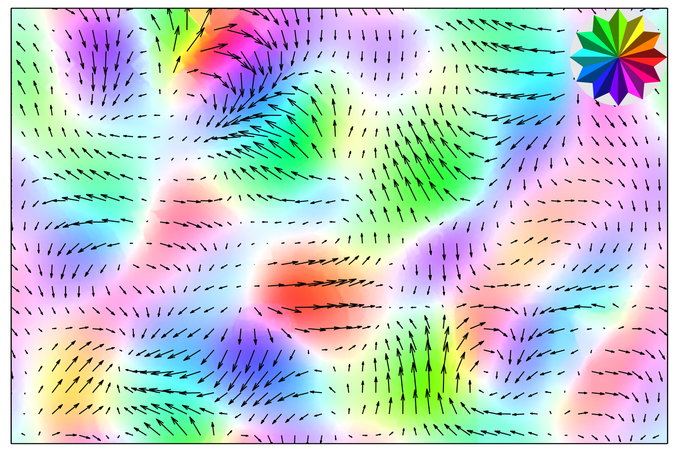
LAP [1] | median error $= 0.01$ pixel, mean error $= 0.102$ pixel, comp. time $= 6$ s



HS [2] | median error $= 0.604$ pixel, mean error $= 0.868$ pixel, comp. time $= 47$ s



LDOF [3] | median error $= 0.701$ pixel, mean error $= 1.310$ pixel, comp. time $= 30$ s



MPOF [4] | median error $= 0.623$ pixel, mean error $= 0.964$ pixel, comp. time $= 279$ s

Approximation order

Using Fourier variables, standard and LAP approximations of the brightness constancy equation $I_2(\mathrm{r}) = I_1(\mathrm{r} - \mathrm{u}(\mathrm{r}))$ can be seen as resulting from the **approximation of the exponential function** by, either a polynomial, or a fraction of polynomials (Padé)

$$I_2(\mathrm{r}) = I_1(\mathrm{r} - \mathrm{u}(\mathrm{r})) = \frac{1}{4\pi^2} \int \hat{I}_1(\boldsymbol{\omega}) \underbrace{\mathrm{e}^{-j\mathrm{u}(\mathrm{r})^{\mathrm{t}}\boldsymbol{\omega}}}_{\text{to be approximated}} \mathrm{e}^{j\mathrm{r}^{\mathrm{t}}\boldsymbol{\omega}} \,\mathrm{d}\boldsymbol{\omega}$$

Optical flow equation $\quad \mathrm{e}^{-j\mathrm{u}(\mathrm{r})^{\mathrm{t}}\boldsymbol{\omega}} = 1 - j\mathrm{u}(\mathrm{r})^{\mathrm{t}}\boldsymbol{\omega} + \mathrm{O}\big(|\mathrm{u}(\mathrm{r})^{\mathrm{t}}\boldsymbol{\omega}|^2\big)$

LAP flow equation $\quad \mathrm{e}^{-j\mathrm{u}(\mathrm{r})^{\mathrm{t}}\boldsymbol{\omega}} = \dfrac{P_n(-j\mathrm{u}(\mathrm{r})^{\mathrm{t}}\boldsymbol{\omega})}{P_n(j\mathrm{u}(\mathrm{r})^{\mathrm{t}}\boldsymbol{\omega})} + \mathrm{O}\big(|\mathrm{u}(\mathrm{r})^{\mathrm{t}}\boldsymbol{\omega}|^{2n+1}\big)$

Theorem. *Consider a location* r_0 *and the local all-pass filter* $h_{\mathrm{r}_0}(\mathrm{r}) = p_{\mathrm{r}_0}(\mathrm{r}) * p_{\mathrm{r}_0}^{-1}(-\mathrm{r})$ *where*

$$\hat{p}_{\mathrm{r}_0}(\boldsymbol{\omega}) = P_n(-j\mathrm{u}(\mathrm{r}_0)^{\mathrm{t}}\boldsymbol{\omega}) \mathrm{e}^{-\frac{1}{2}\sigma^2\|\boldsymbol{\omega}\|^2}$$

Then, if $I_1(\mathrm{r})$ *is sufficiently regular, we have*

$$I_2(\mathrm{r}) - h_{\mathrm{r}_0}(\mathrm{r}) * I_1(\mathrm{r}) = \mathrm{O}\big(\|\mathrm{u}(\mathrm{r}_0)\|^{2n+1}\big);$$

i.e., this approximation is of order $2n$.

Padé approximants

The continued fraction expansion of the exponential function [5,p.70] provides the order $2n$ Padé approximant:

$$\frac{P_n(x)}{P_n(-x)} = 1 + \cfrac{x}{1 - \cfrac{x}{2 + \cfrac{x}{3 - \cfrac{x}{2 + \cfrac{x}{\ddots\, 2 + \cfrac{x}{2n - 1 - \cfrac{x}{2}}}}}}} = \mathrm{e}^x + \mathrm{O}(x^{2n+1})$$

Another option is to use the induction equation

$$\begin{cases} \varepsilon_0(x) = \mathrm{e}^{jx} - 1, \\ \varepsilon_n(x) = j\displaystyle\int_0^x \varepsilon_{n-1}(\xi)(\mathrm{e}^{j(x-\xi)} - 1)\,\mathrm{d}\xi, \text{ for } n \geq 1. \end{cases}$$

which provides $P_n(x)$ through $\varepsilon_n(x) = P_n(-jx)\mathrm{e}^{jx} - P_n(jx)$.

$P_1(x) = 2 + x \quad \rightsquigarrow$ order 2,

$P_2(x) = 6 + 3x + \dfrac{x^2}{2} \quad \rightsquigarrow$ order 4,

$P_3(x) = 20 + 10x + 2x^2 + \dfrac{x^3}{6} \quad \rightsquigarrow$ order 6 etc.

Discussion

In our current practice, the LAP is used with $n = 1$ (only first order derivatives involved, three basis filters) or $n = 2$ (only first and second order derivatives involved, six basis filters). The approximation order theorem shows that the LAP algorithm is of approximation order 2 or of order 4, significantly higher than the order 1 of the standard optical flow equation.

Moreover, in the case where $n = 1$, the related Padé approximant would suggest a variant of the optical flow equation which, despite using only first derivatives, is of order 2:

$$I_2(\mathrm{r}) + \frac{1}{2}\mathrm{u}(\mathrm{r})^{\mathrm{t}}\nabla I_2(\mathrm{r}) = I_1(\mathrm{r}) - \frac{1}{2}\mathrm{u}(\mathrm{r})^{\mathrm{t}}\nabla I_1(\mathrm{r}) + \mathrm{O}\big(\|\mathrm{u}(\mathrm{r})\|^3\big).$$

Finally, the rational Padé approximation validates the choice of basis of the LAP algorithm: partial derivatives of a symmetric function—a Gaussian function in our case.

References

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