Harmonic Spline Series Representation of Scaling Functions

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ABSTRACT

We present here an explicit time-domain representation of any compactly supported dyadic scaling function as a sum of harmonic splines. The leading term in the decomposition corresponds to the fractional splines, a recent, continuous-order generalization of the polynomial splines.

1. INTRODUCTION

The theory of dyadic wavelet decomposition is entirely based on a basic—*scaling*—function $\varphi(x)$ which is assumed to satisfy good *analytic* (approximation-wise) properties (partition of unity, stability) together with a *geometric* condition: a two-scale relation of the form

$$\varphi(x) = \sum_{k\in\mathbb{Z}} h_k \varphi(2x - k). \tag{1}$$

This relation seems to make it almost impossible to express $\varphi(x)$ using standard functions, with the noteworthy exception of the fractional B-spline case which is obtained when $h_k = 2^{-\alpha}\binom{\alpha+1}{k}$ where α is the degree of the spline.[1] For most standard scaling functions such as Daubechies scaling functions, it is indeed possible to compute the value of $\varphi(x)$ exactly for rational arguments only, but not for irrational values like π.

In this paper, we show that all compactly supported scaling functions, i.e., most classical scaling functions, can be expressed in an harmonic form, similar to a Fourier series decomposition. As a result it is possible to have an evaluation of a scaling function at any point, not only rational. Moreover, this decomposition uncovers new scaling functions that had never been considered before: the *harmonic splines*. The result shown here is a development of a similar decomposition that was initially derived by us in another paper[2]

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2. CENTRAL BASIS FUNCTIONS

In order to derive our harmonic decompostion, we first need to *un-localize* the scaling function $\varphi(x)$. We will show indeed in this section that any compactly supported scaling function $\varphi(x)$ can be expressed as a digitally filtered version of a self-similar, one-sided but non compactly-supported function $\rho(x)$

$$\varphi(x) = \sum_{k\geq 0} p_k \rho(x-k). \tag{2}$$

We will call these "central basis function" by analogy to a similar problem arising in the theory of radial basis functions.

Since $\varphi(x)$ is compactly supported, we will assume with no loss of generality that $h_k = 0$ for $k \notin [0, L]$. Then, we define the function $\rho(x)$ as

$$\rho(x) = \begin{cases} \varphi(x) & \text{if } x \in [0,1[\\ h_0^{-j}\rho(x2^{-j}) & \text{if } x \in [2^{j-1}, 2^j[\text{ for some } j \in \mathbb{N} \\ 0 & \text{if } x < 0. \end{cases} \tag{3}$$

Proposition 1. *$\rho(x)$ is self-similar, i.e., it satisfies the property*

$$\rho(x) = h_0 \rho(2x). \tag{4}$$

Moreover, there exists a sequence of coefficients p_k such that (2) *holds.*

Proof. By construction, $\rho(x)$ satisfies the self-similarity property for $x > 1/2$. Then, because $\varphi(x) = 0$ for $x < 0$, we simply observe that the scaling relation (1) reduces to $\varphi(x) = h_0\varphi(2x)$ when $0 \leq x \leq 1$ where $\rho(x)$ is identified as $\varphi(x)$. Whence the self-similar relation.

Next, we consider the function

$$\rho_0(x) = \sum_{k\geq 0} r_k \varphi(x-k) \qquad \text{such that } \begin{cases} r_0 = 1 \\ r_n = h_0^{-1}\sum_{k\geq 0} h_{n-2k} r_k & \text{for all } n \geq 0. \end{cases}$$

Note that $\rho_0(x)$ is such that $\rho_0(x) = \varphi(x)$ for $x \in [0, 1[$. Using the definition of the coefficients r_k, we easily verify that $\rho_0(x) = h_0\rho_0(2x)$. As a result, $\rho_0(x) = \rho(x)$. Since $\rho_0(x)$ is a filtered version of $\varphi(x-k)$, we finally conclude that the reverse is true as well; that is, $\varphi(x)$ is a filtered version of $\rho(x)$. □

Conversely, it is a simple matter to verify that $\varphi(x)$ defined by (2) automatically satisfies a scaling relation of the form (1). Whether it is always possible to find a localization filter with coefficients p_k such that $\varphi(x)$ is in $\mathbf{L}^2$ is still unknown to us; we surmise, though, that the filter defined by

$$p_k = \int_0^\infty \rho(x) \frac{x^k}{k!} e^{-x}\, dx \qquad \text{for } k \geq 0$$

is a good candidate for this goal.

3. HARMONIC DECOMPOSITION

The self-similar equation (4) is very interesting because, as we observe below, it can be recast into a 1-periodicity condition satisfied by an auxiliary function $u(x)$:

$$u(x) = 2^{-\alpha x}\rho(2^x) \quad \Longrightarrow \quad u(x+1) = u(x)$$

where we have let $\alpha = -\log_2(h_0)$.

As we know, periodic functions of $\mathbf{L}^2$ are equal to their Fourier series decomposition almost everywhere which, in the present case, reads:

$$u(x) = \sum_{n\in\mathbb{Z}} c_n e^{2i\pi nx} \qquad \text{where } c_n = \int_0^1 u(\xi)e^{-2i\pi n\xi}\, d\xi$$

We thus obtain the following expression for the central basis function $\rho(x)$:

$$\rho(x) = \sum_{n\in\mathbb{Z}} c_n x_+^{\alpha+\frac{2i\pi n}{\log_2}} \tag{5}$$

where we have used the definition $x_+ = \max(x, 0)$. Note that we have an exact expression of c_n in terms of $\varphi(x)$, in the case where $\rho(x)$ is obtained from a scaling function $\varphi(x)$ (e.g., Daubechies scaling function):

$$c_n = \frac{1}{\log 2}\int_{1/2}^1 \varphi(x)x^{-\alpha-1-\frac{2i\pi n}{\log_2}}\, dx.$$

The harmonic terms $x_+^{\alpha+\frac{2i\pi n}{\log_2}}$ can be localized using a generalized finite difference filter. We call "harmonic splines" the functions that are obtained through this process; their Fourier transform takes the following expression:

$$\hat{\beta}^{\alpha+\frac{2i\pi n}{\log_2}}(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^{1+\alpha+\frac{2i\pi n}{\log_2}}.$$

Notice that these functions are usually not compactly supported. Conversely, we have the identity:

$$x_+^{\alpha+\frac{2i\pi n}{\log_2}} = \sum_{k\geq 0} \frac{\Gamma(1+k+\alpha+\frac{2i\pi n}{\log_2})}{k!}\beta^{\alpha+\frac{2i\pi n}{\log_2}}(x-k)$$

which provides an explicit expression for the coefficients r_k of the "un-localization" filter of the harmonic spline.

Finally, by putting things together, we obtain the main result of this paper:

THEOREM 3.1. *Every compactly supported scaling function $\varphi(x)$ can be expressed as a sum of harmonic splines:*

$$\varphi(x) = \sum_{k\geq 0}\sum_{n\in\mathbb{Z}} \gamma_{k,n}\beta^{\alpha+\frac{2i\pi n}{\log 2}}(x-k) \tag{6}$$

where the coefficients $\gamma_{k,n}$ are defined by

$$\gamma_{k,n} = c_n \sum_{k'\geq 0} p_{k-k'} \frac{\Gamma(1+k'+\alpha+\frac{2i\pi n}{\log 2})}{k'!}$$

This result is surprising in a number of aspects:

- A sum of scaling functions is usually not a scaling function; here, it is only because of very particular coefficients that the expression (6) is a scaling function—and that it is compactly supported.
- Usually, (generalized) spline functions appear through a convolution bringing regularity and approximation order to scaling functions[3]; here, they appear through an addition, using quantified complex degrees.
- If we truncate (6) over n, we get a function that satisfies the very same two-scale difference equation as $\varphi(x)$; but this approximation is usually not in $\mathbf{L}^1$ because its Fourier transform is not continuous at $\omega = 0$.
- The expression (6) makes it possible to evaluate standard scaling functions at arbitrary—in particular, irrational—values of x; as we noticed in introduction, this is unusual for arbitrary scaling functions.

We show in Fig. 1 the behavior of the decomposition formula (6) when we restrict the summation index n to the finite range $[-N, N]$. Note that the fractal structure becomes more apparent and finer grained as one adds more terms to the expansion. Interestingly, the value $\alpha = -\log_2(h_0)$ (real part of the degree of the harmonic splines) coincides with the Hölder regularity of Daubechies scaling function.

REFERENCES

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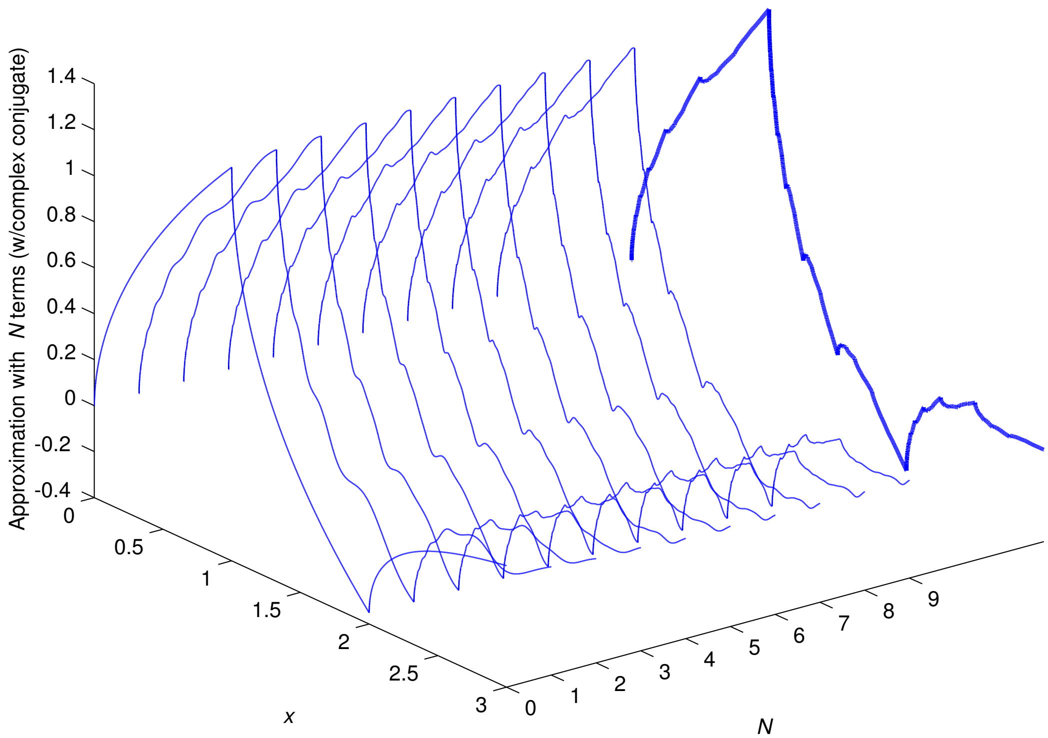

Figure 1. Approximation of Daubechies scaling function of length 4 using the terms c_n in (6) for $|n| \leq N$ and for various values of $N = 0, 1, \ldots 9$. In a bold line, plot of the Daubechies scaling function.