3D Reconstruction of Wave-Propagated Point Sources from Boundary Measurements using Joint Sparsity and Finite Rate of Innovation

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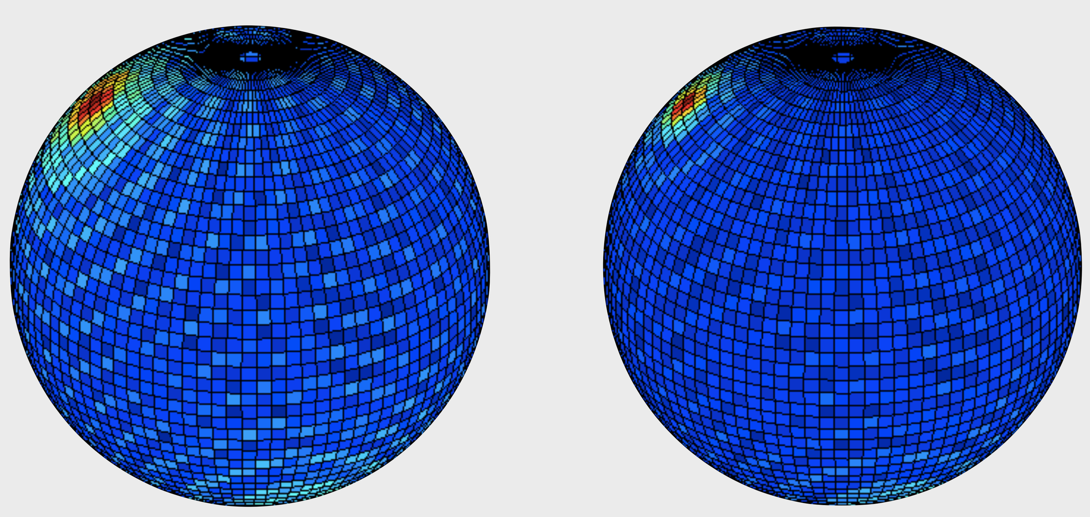
Key points

- In photo/thermo-acoustic imaging, absorbed energy leads to thermoelastic expansion of the tissue that results in a measurable acoustic field on the boundary of the domain. Monitoring the generated acoustic source field yields information about the absorption properties of the tissue.
- Nonionizing laser or RF pulses are used to excite US waves at MHz range. Combining US resolution with high contrast due to EM absorption provides promising applications for soft tissue imaging.
- Recovering the generating source distribution from the boundary measurements is an ill-posed problem that necessitates additional assumptions on the solution. Here, we extend our recently proposed non-iterative reconstruction scheme for point source models with single frequency field measurements to source configurations that shows joint sparsity for different temporal frequency measurements.
- With specific choice of sensing functions, we first extract generalized samples of the innovative signal from the boundary measurements. Then, we jointly annihilate these samples to reconstruct the the projection of the source points in the complex plane. We provide a modified multi-source Dijkstra algorithm to recover the remaining Z-localization and a denoising scheme for model mismatch and noisy measurements.
- Experimental results demonstrate that jointly-sparse reconstruction achieves CRLBs for lower SNR levels than single frequency measurement.

Generalized Sensing

Problem Statement

- Knowing the pressure field only on the boundary by $U|_{\partial\Omega}$ and $\nabla U|_{\partial\Omega}$ find the enclosed acoustic source distribution within Ω.



Acoustic field on the boundary   Normal gradient on the boundary

Helmholtz Equation

- Observing the time harmonic solutions of the wave equation, we obtain,

$$\nabla^2 U + k^2 U = -P$$

Source Modeling

- Reconstructing the source distribution from the boundary measurements is an ill-posed problem.
- Point Source Model is valid under stress-confinement condition

$$P(\mathbf{r}) = \sum_{m=1}^{M} c_m \delta(\mathbf{r} - \mathbf{r}_m)$$

Finite Rate of Innovation

- An FRI signal has two parts: An innovative and a non-innovative.
- Knowing set of functions $\{\phi_r(x)\}_{r=1,\dots,R}$ with arbitrary shifts and weights (innovations), an FRI signal can be written as

$$f(x) = \sum_{n\in\mathbb{Z}} \sum_{r=0}^{R} c_{nr}\phi_r(x - x_n)$$

- Having samples of such a signal twice the innovation rate provides exact reconstruction replacing the nonlinear parameter estimation problem with equivalent two linear system solutions.

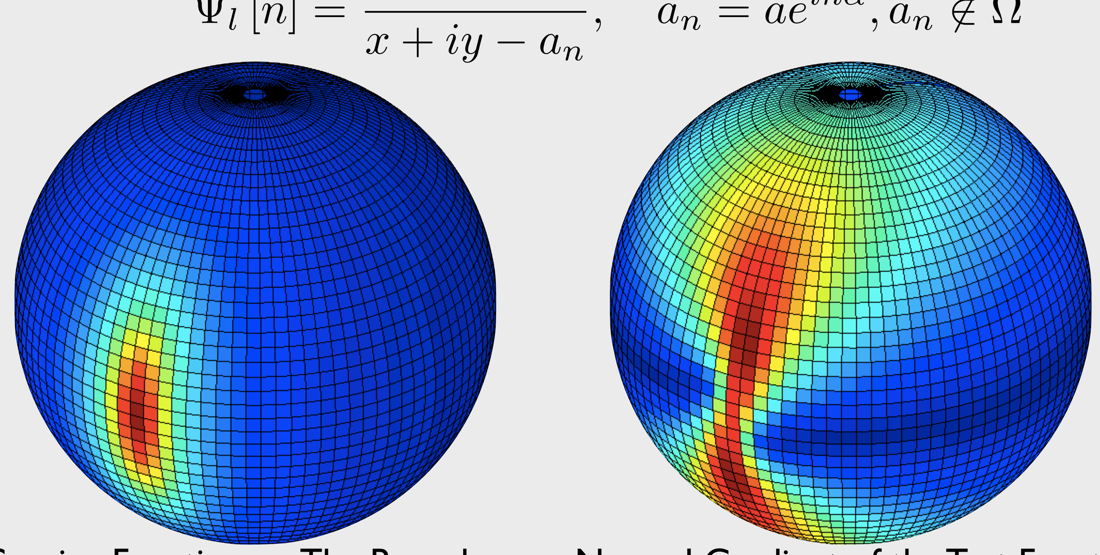
Sensing From Boundary Measurements

- Link between the innovative signal and the measured boundary data through second Green's Identity
- The *generalized-samples* of the source signal can be extracted from the boundary measurements as

$$\langle \Psi, P\rangle = \oint_{\partial\Omega} (\Psi\nabla U - U\nabla\Psi)\,.\mathbf{e}_{\partial\Omega} dS \qquad \nabla^2\Psi = -k^2\Psi$$

Proposed Family of Test Functions

$$\Psi_l[n] = \frac{e^{ik_l z}}{x + iy - a_n}, \quad a_n = a e^{in\alpha}, a_n \notin \Omega$$



Sensing Function on The Boundary   Normal Gradient of the Test Function

Jointly Sparse Source Recovery

Generalized Samples

- The samples of the innovative source signal satisfy

$$\mu_l[n] = \langle \Psi_l[n], P\rangle = \sum_{m=1}^{M} \frac{c_m e^{ik_l z_m}}{s_m - a_n} = \frac{\sum_{m=0}^{M-1} c'_m e^{imn\alpha}}{\prod_{m=1}^{M}(s_m - a_n)}$$

for the selected wave number where the poles of sensing function lie at equidistant angles on the complex plane.

Annihilation Filter

- The exponential terms in the denominator can be annihilated by

$$H(z) = \prod_{r=0}^{M-1}(1 - e^{ir\alpha}z^{-1}) = \sum_{r=0}^{M} h_r z^{-r}$$

XY-Plane Projection

- Defining a polynomial whose roots are the XY-plane projections

$$X(a_n) = \prod_{m=1}^{M}(s_m - a_n) = \sum_{q=0}^{M} x_q a_n^q \text{ with } x_M = 1$$

the annihilation equation has the following form

$$\{\mu_l[\cdot]X(a.)\} * h = 0, \text{ for all } l \in [\![1, L]\!]$$

which can be formulated in matrix representation as

$$\mathbf{A}_l\mathbf{x} = \mathbf{H}\mathbf{D}_l\mathbf{V}\mathbf{x} = \mathbf{0}$$

$$\underbrace{\begin{bmatrix} h_M & .. & h_0 & .. & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & .. & h_M & .. & h_0 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} \mu_l[0] & 0 & \cdots \\ 0 & \ddots & \vdots \\ \vdots & \dots & \mu_l[N-1] \end{bmatrix}}_{\mathbf{D}_l} \underbrace{\begin{bmatrix} a_0^0 & \cdots & a_0^M \\ \vdots & \vdots & \vdots \\ a_{N-1}^0 & \dots & a_{N-1}^M \end{bmatrix}}_{\mathbf{V}}$$

Number of Minimum Generalized Samples

- For the noiseless case, the linear system has rank M with (N-M) equations and M unknowns.
- We need at least N=2M generalized samples of the signal which is twice the innovation rate for complex plane projection.

Joint Annihilation

- We Extend the above system to incorporate multiple generalized samples taken at different frequencies

$$\mathbf{H}\mathbf{D}_l\mathbf{V}\mathbf{x} = 0 \quad \vdots \quad \mathbf{H}\mathbf{D}_L\mathbf{V}\mathbf{x} = 0$$

Model Mismatch and Denoising

- Assuming the generalized samples are corrupted with cAWGN

$$\hat{\mu}_l[n] = \mu_l[n] + v[n]$$

which changes the rank of the linear system,

- Preconditioning: Use the unitary matrices
  - Find low-rank approximation (1)
  - Extract the denoised samples minimizing (2)

Algorithm 1: Cadzow-like Denoising

Data: $\hat{\mathbf{D}}_l$,(Corrupted with cAWGN) and assume Higher Number of Source, $\tilde{M} > M$
Result: Denoised Generalized Samples $\mathbf{D}_l$
begin
  $\tilde{\mathbf{A}}_{l0} \longleftarrow \tilde{\mathbf{H}}_0\hat{\mathbf{D}}_l\tilde{\mathbf{V}}_0$;
  while $rank(\tilde{\mathbf{A}}_{l0}) > M$ **do**
    $\hat{\mathbf{A}} \longleftarrow \underset{rank(\hat{\mathbf{A}})=M}{\operatorname{argmin}} ||\hat{\mathbf{A}} - \tilde{\mathbf{A}}_{l0}||_F$ (1)
    $\mathbf{D}_l \longleftarrow \underset{\mathbf{D}}{\operatorname{argmin}} ||\tilde{\mathbf{H}}_0\mathbf{D}\tilde{\mathbf{V}}_0 - \hat{\mathbf{A}}||_F$ (2)
    $\tilde{\mathbf{A}}_{l0} \longleftarrow \tilde{\mathbf{H}}_0\mathbf{D}_l\tilde{\mathbf{V}}_0$;

Magnitude and Z-Axis Estimation

Magnitude Recovery

- With the same set of generalized samples and estimated XY-locations, we solve for the the magnitudes for each frequency

$$\mu_l[n] = \sum_{m=1}^{M} \frac{c_m e^{ik_l z_m}}{s_m - a_n}, \; n \in [\![1, N]\!]$$

which can be formulated in matrix representation as

$$\underbrace{\begin{bmatrix} \frac{1}{s_1 - a_1} & \cdots & \frac{1}{s_M - a_1} \\ \vdots & \vdots & \vdots \\ \frac{1}{s_1 - a_N} & \cdots & \frac{1}{s_M - a_1} \end{bmatrix}}_{\mathbf{E}} \underbrace{\begin{bmatrix} c_1 e^{ik_l z_1} \\ \vdots \\ c_M e^{ik_l z_M} \end{bmatrix}}_{\mathbf{f}_l} = \underbrace{\begin{bmatrix} \mu_l(1) \\ \vdots \\ \mu_l(N) \end{bmatrix}}_{=\mu_l^T}$$

Periodicity of the Z Locations inherent the sensing function

- Due to exponential term in the sensing function, we have a set of periodic solutions for each source point.

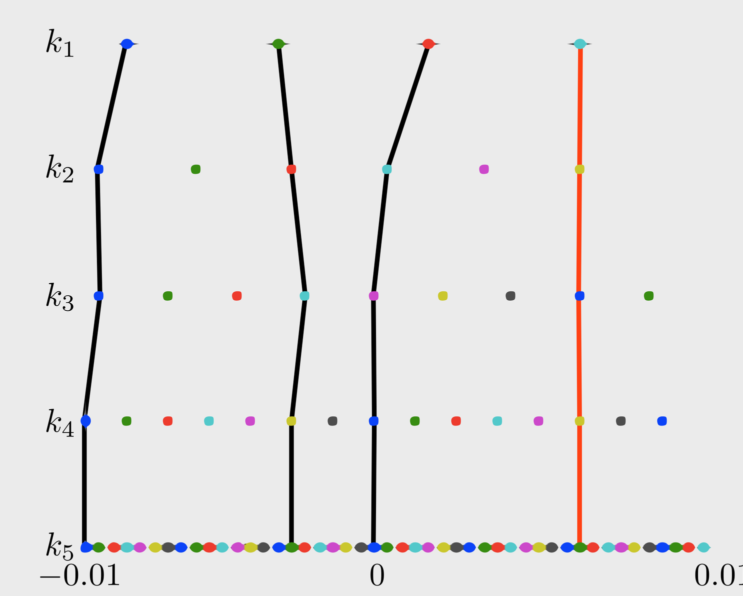
$$\mathbf{Z}(l) = \begin{bmatrix} z_1(l) \\ \vdots \\ z_M(l) \end{bmatrix} = \frac{\arg(\mathbf{f}_l) + 2\pi n_l}{k_l}, \; \forall \begin{cases} n_l \in \mathbb{Z} \\ z_m(l) \in \Omega \\ l = 1 : L \end{cases}$$

Modified Dijkstra's Algorithm

- Multi-source Dijkstra's algorithm for closed shortest path.

$$\mathbf{Z} = \begin{bmatrix} z_1 \\ \vdots \\ z_M \end{bmatrix} = \frac{1}{L}\sum_{l=1}^{L} \frac{\arg(\mathbf{f}_l) + 2\pi\hat{n}_l}{k_l}$$

$$\hat{n}_l = \underset{l}{\operatorname{argmin}} \sum_{l=2}^{L} \left| \frac{\arg(\mathbf{f}_1(m)) + 2\pi n_1}{k_1} - \frac{\arg(\mathbf{f}_l(m)) + 2\pi n_l}{k_l} \right|$$

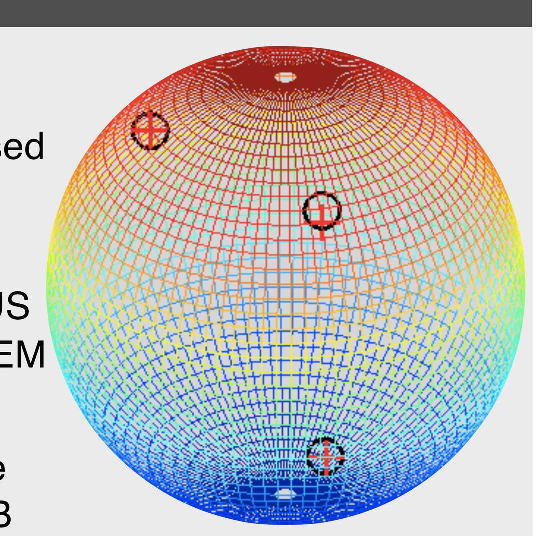


Illustrative Example for Dijkstra's Method: Red is the minimum distance path

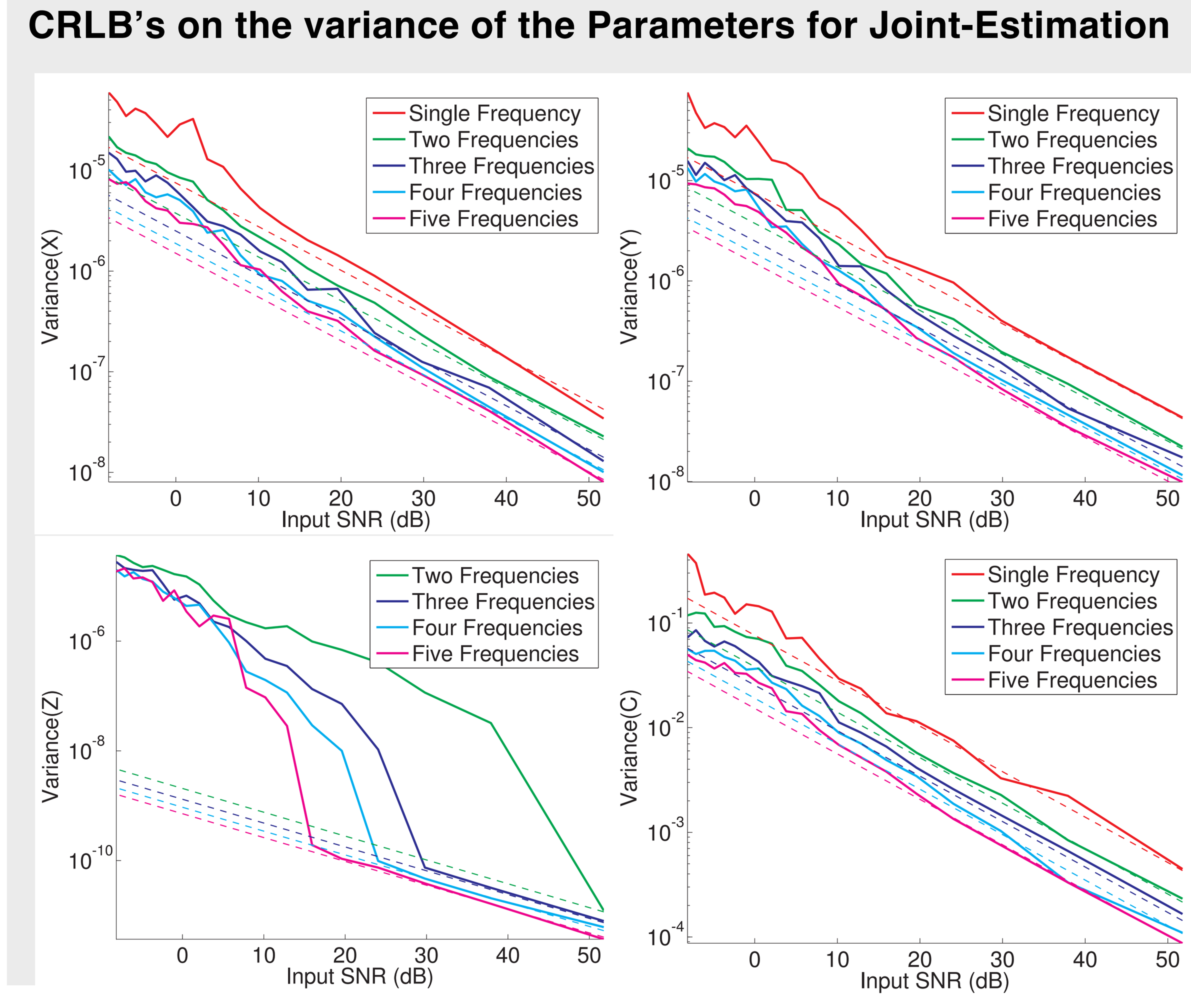
Application: Photoacoustic Imaging

Hybrid Source Imaging Modality

- Novel hybrid imaging technique based on generation of acoustic waves by absorption of EM energy
- The motivation is to combine high US resolution with high contrast due to EM absorption
- Localization of PA generated source points within a radius of 1cm at 20dB on the generalized samples



CRLB's on the variance of the Parameters for Joint-Estimation



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