Random Variables

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engg2040c tutorial 9

Outline

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Brief Review

- Binomial random variable:

$$p(i) = \binom{n}{k} p^i (1-p)^{n-i}$$

- Poisson random variable:

$$p(i) = e^{-\lambda} \frac{\lambda^i}{i!}$$

- Geometric random variable:

$$p(i) = p(1-p)^i$$

- Negative Binomial random variable:

$$p(i) = \binom{i-1}{r-1} p^r (1-p)^{i-r}$$

Brief Review

- Hyper-Geometric random variable:

$$p(i) = \frac{\binom{m}{i}\binom{N-m}{n-i}}{\binom{N}{n}}$$

- The expected value of a sum of random variables is equal to the sum of their expected values

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$$

- Indicator random variable for event A:

$$I = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Outline

Balls and Bins

Problem description

- Given m balls and n bins
- Each ball is throw to a random bin

Question: what is the expected number of empty bins?

Hint: Indicator random variable & $E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i]$

Solution

- Let Y_i be an indicator random variable such that

$$Y_i = \begin{cases} 1 & \text{if event bin } i \text{ is empty} \\ 0 & \text{otherwise} \end{cases}$$

- Then $Y = \sum_{i=1}^{n} Y_i$ is the random variable that denotes the number of empty bins
- Since $E[Y] = E[\sum_{i=1}^{n} Y_i] = \sum_{i=1}^{n} E[Y_i]$, we can compute $E[Y]$ via computing $E[Y_i]$

Solution

$$P\{Y_i = 1\} = (1 - \frac{1}{n})^m$$

$$\begin{aligned} E[Y_i] &= 1 \times P\{Y_i = 1\} + 0 \times P\{Y_i = 0\} \\ &= (1 - \frac{1}{n})^m \end{aligned}$$

$$\begin{aligned} E[Y] &= \sum_{i=1}^{n} E[Y_i] \\ &= \sum_{i=1}^{n} (1 - \frac{1}{n})^m \\ &= n(1 - \frac{1}{n})^m \end{aligned}$$

Outline

Check payment

Problem description

- Three friends go for coffee, they decide who will pay the check by each flipping a coin
- They let the "odd" person pay
- If all three flips produce the same result, then make a second round on flips, they continue to do so until there is an odd person

Question: what is the probability that

1. exactly 3 rounds of flips are made?
2. more than 4 rounds are needed?

Solution

- The probability that a round does not result in an "odd person" is the probability that all three flips are the same, Thus

$$P\{\text{a round does not result in an "odd person"}\} = \frac{2}{2^3} = \frac{1}{4}$$

Solution

1. Exactly 3 rounds will be made, if the first 2 rounds do not result in "odd person" and the third round results in an "odd person", Thus

$$P\{\text{exactly 3 rounds of flips are made}\} = (\frac{1}{4})^2(1-\frac{1}{4})$$

2. More than 4 rounds will be made, if the first 4 rounds do not result in "odd person", Thus

$$P\{\text{more than 4 rounds of flips are made}\} = (\frac{1}{4})^4$$

Outline

Play a game

Problem description

- An urn initially has N white and M black balls
- Balls are randomly withdrawn, one at a time without replacement

Question: find the probability that n white balls are drawn before m black balls, $n < N, m < M$

Solution

- A total of n balls will be withdrawn before a total m black balls if and only if there are at least n white balls in the first $n+m-1$ withdrawals.
- Let X denote the number of white balls among the first $n+m-1$ balls withdrawn, then X is a hypergeometric random variable, and it follows that

$$P[X \geq n] = \sum_{i=n}^{n+m-1} P\{X = i\} = \sum_{i=n}^{n+m-1} \frac{\binom{N}{i}\binom{M}{n+m-1-i}}{\binom{N+M}{n+m-1}}$$

Outline

Play a game

Problem description

- You play a game repeatedly and each time you win with probability p
- You plan to play 5 times, but if you win in the fifth time, then you will keep on playing until you lose

Question:

1. Find the expected times that you play?
2. Find the expected times that you lose?

Solution

Let X denote the times that you play and Y denote the times that you lose

- The probability that a round does not result in an "odd person" is the probability that all three flips are the same, Thus

$$P\{\text{a round does not result in an "odd person"}\} = \frac{2}{2^3} = \frac{1}{4}$$

Solution

1. After your fourth play, you will continue to play until you lose. Therefore, $X-4$ is a geometric random variable with parameter $1-p$, so

$$E[X] = E[4+(X-4)] = 4 + E[X-4] = 4 + \frac{1}{1-p}$$

2. Let Z denote the number of losses in the first 4 games, then Z is a binomial random variable with parameters 4 and $1-p$. Because $Y = Z+1$, so we have

$$E[Y] = E[Z+1] = E[Z] + 1 = 4(1-p) + 1$$

Outline

Play games between two teams

Problem description

- Teams A and B play a series of games
- The winner is the first team winning 3 games
- Suppose that team A independently wins each game with probability p

Question: What is the probability that A wins

1. the series given that it wins the first game?
2. the first game given that it wins the series?

Solution

1. Given that A wins the first game, it will win the series if, from then on, it wins 2 games before team B wins 3 games.

- Thus, in the following four games, A must win at least 2 games to win the game. Thus

$$P\{A \text{ wins} | A \text{ wins first game}\} = \sum_{i=2}^{4} \binom{4}{i} p^i (1-p)^{4-i}$$

Solution

- Let p denote the probability that A wins the first game given that it wins the series is:

$$\begin{aligned} p &= P\{A \text{ wins first game} | A \text{wins}\} \\ &= \frac{P\{A \text{ wins, } A \text{ wins first game}\}}{P\{A \text{ wins}\}} \\ &= \frac{P\{A \text{ wins} | A \text{ wins first game}\} P\{A \text{ wins first game}\}}{P\{A \text{ wins}\}} \\ &= \frac{\sum_{i=2}^{4} \binom{4}{i} p^{i+1} (1-p)^{4-i}}{\sum_{i=3}^{5} \binom{5}{i} p^{i} (1-p)^{5-i}} \end{aligned}$$

Outline

Play a game

Problem description

- X and Y are two independent random variables
- X is a Poisson random variable with parameter λ_1
- Y is a Poisson random variable with parameter λ_2

Show that $Z = X + Y$ is a Poisson random variable with parameter $\lambda_1 + \lambda_2$

Solution

$$
\begin{aligned}
P\{Z=i\} &= P\{X+Y=i\} \\
&= \sum_{k=0}^{i} P\{Y=k, X=i-k\} \\
&= \sum_{k=0}^{i} P\{Y=k\}P\{X=i-k\} \\
&= \sum_{k=0}^{i} (e^{-\lambda_2}\frac{\lambda_2^k}{k!})(e^{-\lambda_1}\frac{\lambda_1^{i-k}}{(i-k)!}) \\
&= \sum_{k=0}^{i} e^{-(\lambda_1+\lambda_2)}\frac{\lambda_2^k \lambda_1^{i-k}}{k!(i-k)!}
\end{aligned}
$$

Solution

$$
\begin{aligned}
P\{Z = i\} &= \sum_{k=0}^{i} e^{-(\lambda_1+\lambda_2)} \frac{i!\lambda_2^k \lambda_1^{i-k}}{i!k!(i-k)!} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{i!} \sum_{k=0}^{i} \frac{i!}{k!(i-k)!} \lambda_2^k \lambda_1^{i-k} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{i!} \sum_{k=0}^{i} \binom{i}{k} \lambda_2^k \lambda_1^{i-k} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{i!} (\lambda_1 + \lambda_2)^i \\
&= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1 + \lambda_2)^i}{i!}
\end{aligned}
$$

Thank You!