DFA Minimization, Pumping Lemma

CSCI 3130 Formal Languages and Automata Theory

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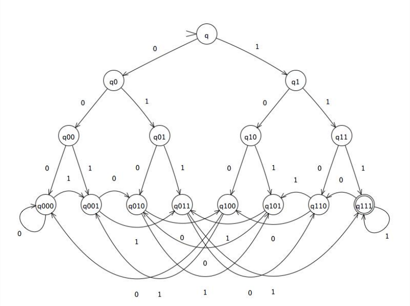
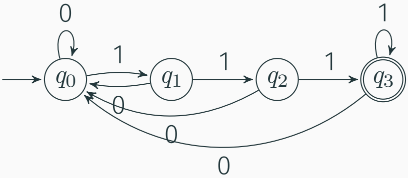
Chinese University of Hong Kong

$L =$ strings ending in 111





Can we do it in 3 states?

Even smaller DFA?

$L =$ strings ending in 111

Intuitively, needs to remember number of ones recently read

We will show

$\varepsilon, 1, 11, 111$ are pairwise distinguishable by L

In other words

$(\varepsilon, 1), (\varepsilon, 11), (\varepsilon, 111), (1, 11), (1, 111), (11, 111)$

are all distinguishable by L

Then use this result from last lecture:

If strings $x_1, \ldots, x_n$ are pairwise distinguishable by L, any DFA accepting L must have at least n states

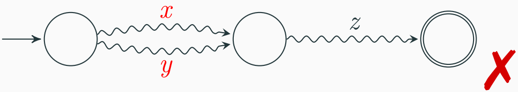
Recap: distinguishable strings

What do we mean by "1 and 11 are distinguishable"?

(x, y) are distinguishable by L if there is string z such that $xz \in L$ and $yz \notin L$ (or the other way round)

We saw from last lecture

If x and y are distinguishable by L, any DFA accepting L must reach different states upon reading x and y



Distinguishable strings

Why are 1 and 11 distinguishable by L?

$L =$ strings ending in 111

Take $z = 1$

$11 \notin L \qquad 111 \in L$

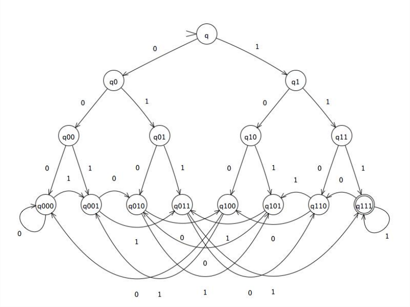
More generally, why are 1^i and 1^j distinguishable by L?
$(0 \leqslant i < j \leqslant 3)$

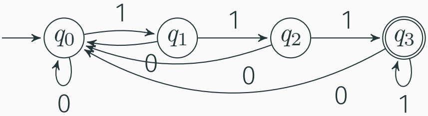
Take $z = 1^{3-j}$

$1^i 1^{3-j} \notin L \qquad 1^j 1^{3-j} \in L$

$\varepsilon, 1, 11, 111$ are pairwise distinguishable by L

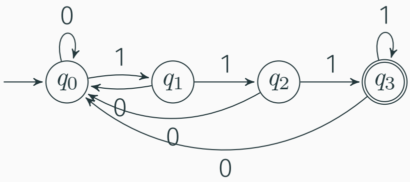
Thus our 4-state DFA is minimal





We now show how to turn any DFA for L into the minimal DFA for L

Minimal DFA and distinguishability



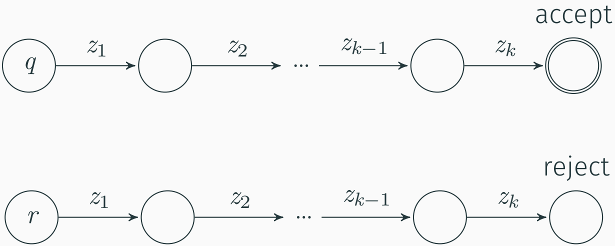
Distinguishable strings must be in different states

Indistinguishable strings may end up in the same state

DFA minimial $\Leftrightarrow$ Every pair of distinct states is distinguishable

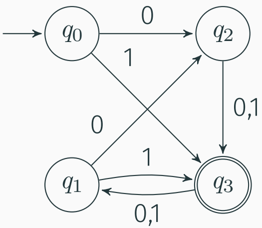
Distinguishable states

Two states q and r are distinguishable if



on the same continuation string $z = z_1 \ldots z_k$,
one accepts, but the other rejects

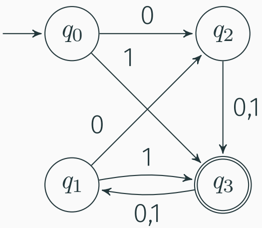
Examples of distinguishable states



Which of the following pairs are distinguishable? by which string?

(q_0, q_3)
(q_1, q_3)
(q_2, q_3)
(q_1, q_2)
(q_0, q_2)
(q_0, q_1)

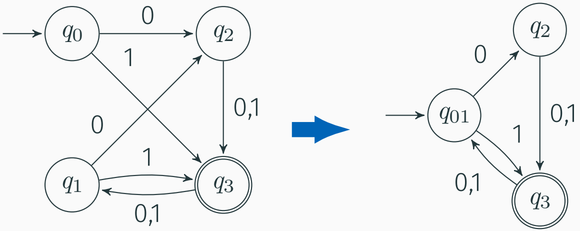
Examples of distinguishable states



Which of the following pairs are distinguishable? by which string?

(q_0, q_3) distinguishable by ε
(q_1, q_3) distinguishable by ε
(q_2, q_3) distinguishable by ε
(q_1, q_2) distinguishable by 0
(q_0, q_2) distinguishable by 0
(q_0, q_1) indistinguishable

Examples of distinguishable states



Which of the following pairs are distinguishable? by which string?

(q_0, q_3) distinguishable by ε
(q_1, q_3) distinguishable by ε
(q_2, q_3) distinguishable by ε
(q_1, q_2) distinguishable by 0
(q_0, q_2) distinguishable by 0
(q_0, q_1) indistinguishable

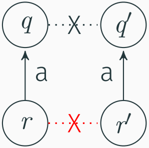
indistinguishable pairs
can be merged

Finding (in)distinguishable states

Phase 1:

For each accepting q and rejecting q'
Mark (q, q') as distinguishable (X)

Phase 2:

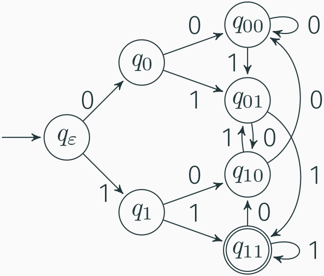


If (q, q') are marked and
$r \xrightarrow{a} q \quad r' \xrightarrow{a} q'$
Mark (r, r') as distinguishable (X)

Phase 3:

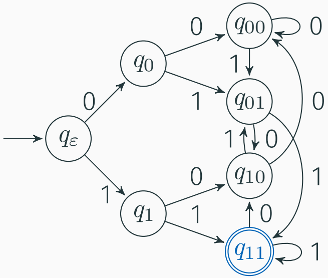
Unmarked pairs are indistinguishable
Merge them into groups

DFA minimization: example



q_0						
q_1						
q_{00}						
q_{01}						
q_{10}						
q_{11}						
	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}

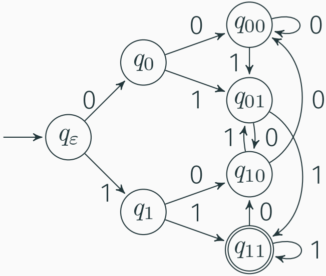
DFA minimization: example



	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}
q_0						
q_1						
q_{00}						
q_{01}						
q_{10}						
q_{11}	X	X	X	X	X	X

(Phase 1) q_{11} is distinguishable from all other states

DFA minimization: example

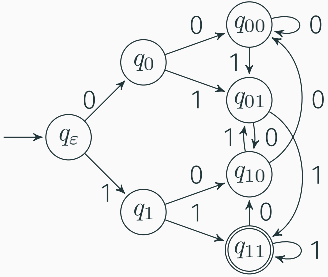


q_0						
q_1						
q_{00}						
q_{01}						
q_{10}						
q_{11}	X	X	X	X	X	X
	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}

(Phase 2) Looking at $(r, r') = (q_\varepsilon, q_0)$

Neither (q_0, q_{00}) nor (q_1, q_{01}) are distinguishable

DFA minimization: example

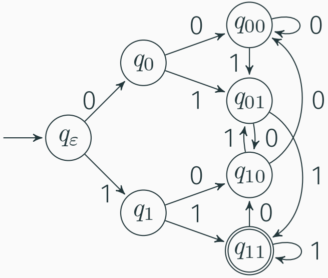


	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}
q_0						
q_1	X					
q_{00}						
q_{01}						
q_{10}						
q_{11}	X	X	X	X	X	X

(Phase 2) Looking at $(r, r') = (q_\varepsilon, q_1)$

(q_1, q_{11}) is distinguishable

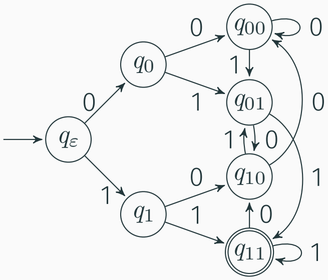
DFA minimization: example



	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}
q_0						
q_1	X	X				
q_{00}			X			
q_{01}	X	X		X		
q_{10}			X		X	
q_{11}	X	X	X	X	X	X

(Phase 2) After going through the whole table once
Now we make another pass

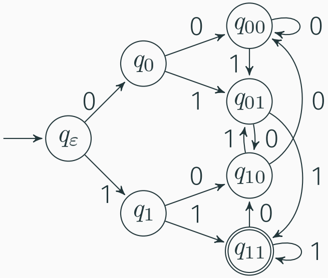
DFA minimization: example



	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}
q_0						
q_1	X	X				
q_{00}			X			
q_{01}	X	X		X		
q_{10}			X		X	
q_{11}	X	X	X	X	X	X

(Phase 2) Looking at $(r, r') = (q_\varepsilon, q_0)$
Neither (q_0, q_{00}) nor (q_1, q_{01}) are distinguishable

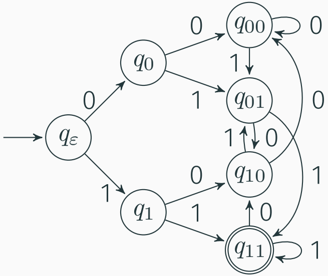
DFA minimization: example



	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}
q_0						
q_1	X	X				
q_{00}			X			
q_{01}	X	X		X		
q_{10}			X		X	
q_{11}	X	X	X	X	X	X

(Phase 2) Looking at $(r, r') = (q_\varepsilon, q_{00})$

Neither (q_0, q_{00}) nor (q_1, q_{01}) are distinguishable



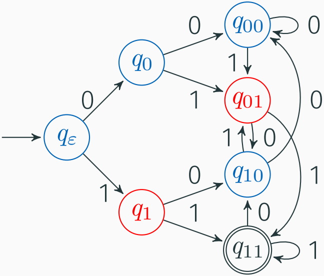
	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}
q_0						
q_1	X	X				
q_{00}			X			
q_{01}	X	X		X		
q_{10}			X		X	
q_{11}	X	X	X	X	X	X

(Phase 2) Nothing changes in the second pass

Ready to go to Phase 3

Now every unmarked pair is indistinguishable

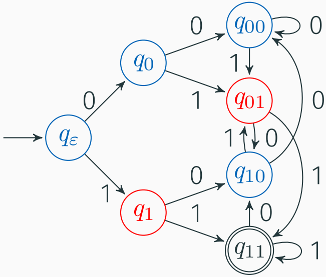
DFA minimization: example



	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}
q_0	A					
q_1	X	X				
q_{00}	A	A	X			
q_{01}	X	X	B	X		
q_{10}	A	A	X	A	X	
q_{11}	X	X	X	X	X	X

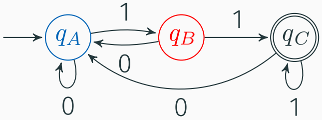
(Phase 3) Merge indistinguishable states into groups
(also known as equivalence classes)

DFA minimization: example



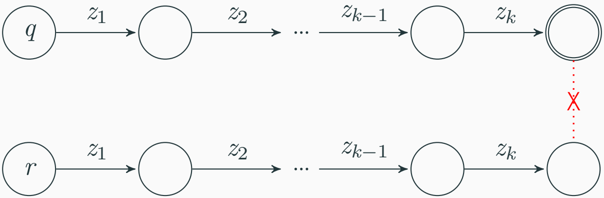
	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}
q_0	A					
q_1	X	X				
q_{00}	A	A	X			
q_{01}	X	X	B	X		
q_{10}	A	A	X	A	X	
q_{11}	X	X	X	X	X	X

Minimized DFA:



Why it works

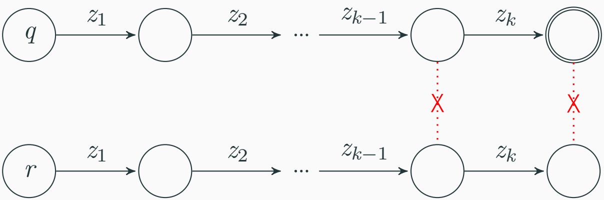
Why have we found all distinguishable pairs?



Because we work backwards

Why it works

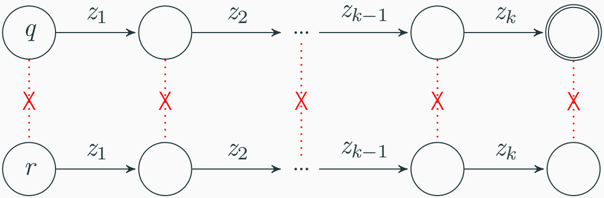
Why have we found all distinguishable pairs?



Because we work backwards

Why it works

Why have we found all distinguishable pairs?



Because we work backwards

Pumping Lemma

Pumping lemma

Another way to show some language is irregular

Example

$L = \{0^n 1^n \mid n \geqslant 0\}$ is irregular

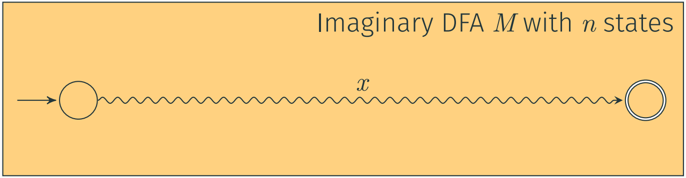
We reason by contradiction:

Suppose we have a DFA M for L

Something must be wrong with this DFA

M must accept some strings outside L

Towards a contradiction



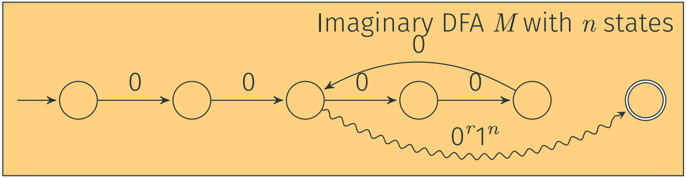
What happens when M gets input $s = 0^n1^n$?

M accepts s because $s \in L$

M has n states, it must revisit one of its states while reading 0^n

(i.e. first n symbols of x)

Towards a contradiction



What happens when M gets input $s = 0^n1^n$?

M accepts s because $s \in L$

M has n states, it must revisit one of its states while reading 0^n

(i.e. first n symbols of x)

The DFA must contain a cycle consisting of 0's

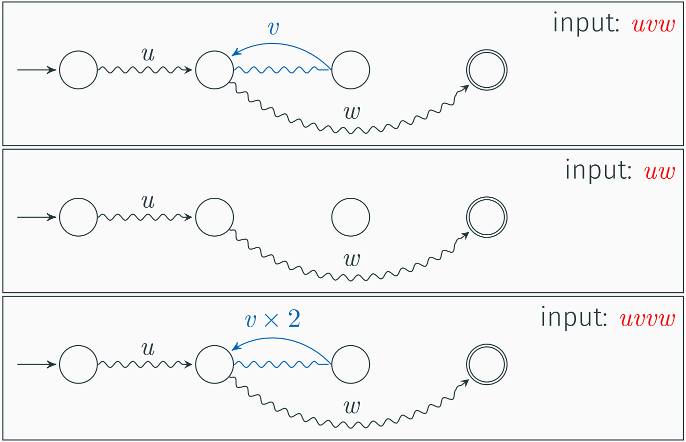
M will also accept strings that go around the cycle multiple times

But such strings have more 0s than 1s and cannot be in L

Copy & pasting the cycle yields accepting paths

Split s into uvw

$u =$ before the first cycle $\quad v =$ first cycle $\quad w =$ the rest



Proof structure

I suspect $L = \{0^n 1^n \mid n \geqslant 0\}$ may be regular

No way! How many states are there in your DFA accepting L?

My DFA has $n = 4$ states

Your DFA must accept $s = 0^n 1^n$

The DFA will go through a cycle during the first n symbols of s

Split $s = uvw$ ($u =$ before first cycle, $v =$ first cycle)

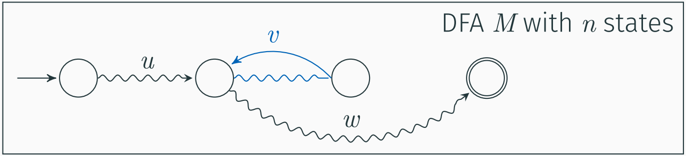
Your DFA erroneously accepts $uw = 0^m 1^n$ ($m < n$)

You're right

Pumping lemma for regular languages

For every regular language L, there exists a number n such that
for every string $s \in L$ longer than n symbols,
we can write $s = uvw$ where

1. $|uv| \leqslant n$
2. $|v| \geqslant 1$
3. For every $i \geqslant 0$, the string uv^iw is in L



$n =$ number of states in imaginary DFA M for L

$i =$ number of times to go around the first cycle

Proving languages are irregular

For every regular language L, there exists a number n such that
for every string $s \in L$ longer than n symbols,
we can write $s = uvw$ where

1. $|uv| \leqslant n$
2. $|v| \geqslant 1$
3. For every $i \geqslant 0$, the string $uv^i w$ is in L

To show that a language L is irregular
we need to find arbitrarily long s in L
so that no matter how the lemma splits s into u, v, w (subject to $|uv| \leqslant n$ and $|v| \geqslant 1$)
we can find $i \geqslant 0$ such that $uv^i w \notin L$

Example

$$L_2 = \{0^m 1^n \mid m > n \geqslant 0\}$$

1. For any n (number of states of an imaginary DFA accepting L_2)
2. There is a string $s = 0^{n+1}1^n$ in L_2
3. Pumping lemma splits s into uvw $\quad (|uv| \leqslant n$ and $|v| \geqslant 1)$
4. Choose $i = 0$ so that $uv^iw \notin L_2$

Example: 00000011111