Pushdown automata

CSCI 3130 Formal Languages and Automata Theory

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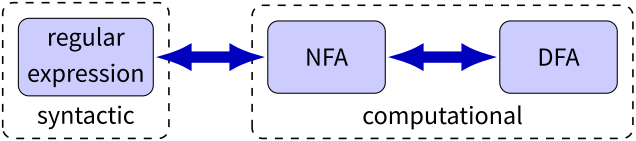
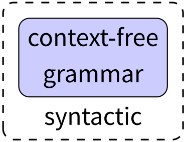
Chinese University of Hong Kong

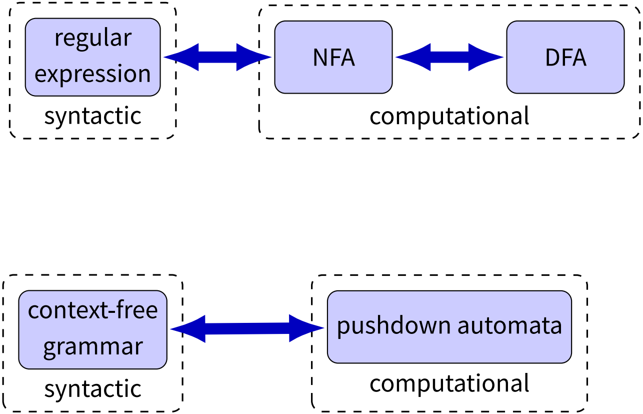
Fall 2016

Syntax vs computation

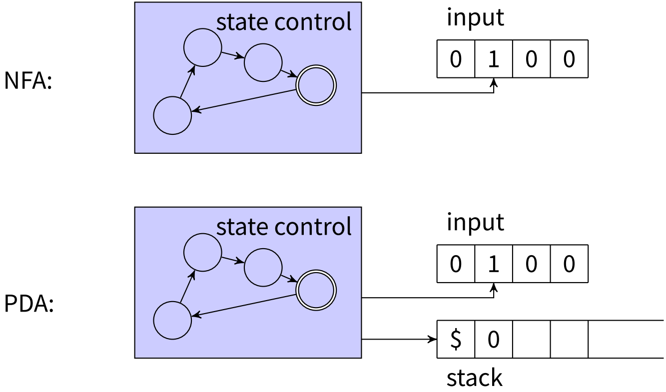




Syntax vs computation

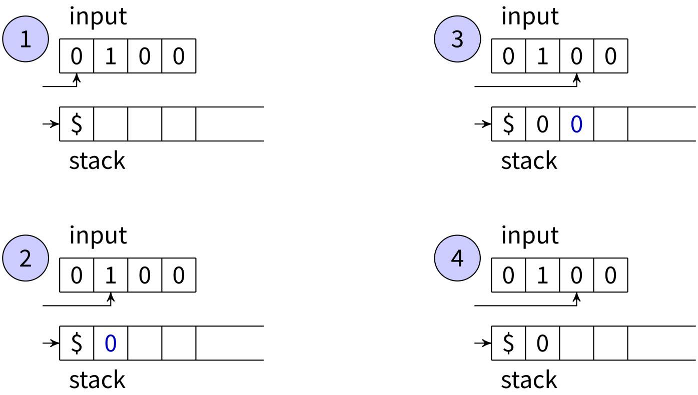


NFA vs pushdown automaton



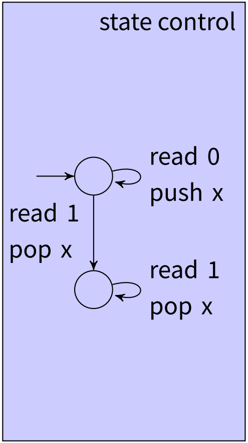
A pushdown automaton (PDA) is like an NFA but with an infinite stack

Pushdown automata



As the PDA reads the input, it can push/pop symbols from the top of the stack

Building a PDA



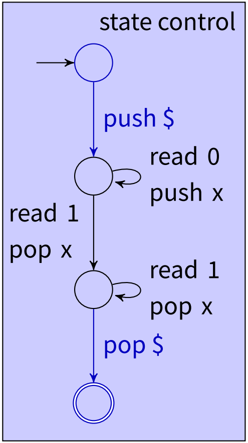
$$L = \{0^n 1^n \mid n \geqslant 1\}$$

Remember each 0 by pushing x onto the stack

Upon reading a 1, pop x from the stack

We want to accept when the hit the stack bottom

Building a PDA



$$L = \{0^n 1^n \mid n \geqslant 1\}$$

Remember each 0 by pushing x onto the stack

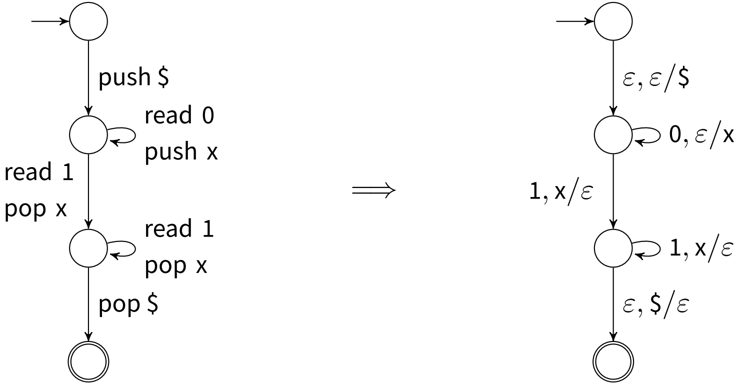
Upon reading a 1, pop x from the stack

We want to accept when the hit the stack bottom

Use $ to mark the stack bottom

Example input: 000111

Notation for PDAs



read a, pop b / push c

If next symbol is a and top of stack is b, may pop b and push c

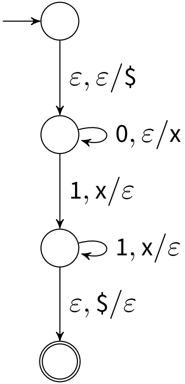
Definition of PDA

A pushdown automaton is $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where:

- Q is a finite set of states
- Σ is the input alphabet
- Γ is the stack alphabet
- $q_0 \in Q$ is the initial state
- $F \subseteq Q$ is the set of accepting states
- δ is the transition function

$$\delta : \underbrace{Q}_{\text{state}} \times \underbrace{(\Sigma \cup \{\varepsilon\})}_{\text{input symbol}} \times \underbrace{(\Gamma \cup \{\varepsilon\})}_{\text{pop symbol}} \to \text{subsets of} \left\{ \underbrace{Q}_{\text{state}} \times \underbrace{(\Gamma \cup \{\varepsilon\})}_{\text{push symbol}} \right\}$$

Example



$$\Sigma = \{0, 1\}$$
$$\Gamma = \{\$, x\}$$

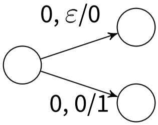
$$\delta(q_0, \varepsilon, \varepsilon) = \{(q_1, \$)\}$$
$$\delta(q_0, \varepsilon, \$) = \emptyset$$
$$\delta(q_0, \varepsilon, x) = \emptyset$$
$$\delta(q_0, 0, \varepsilon) = \emptyset$$
$$\vdots$$

$$\delta : \underbrace{Q}_{\text{state}} \times \underbrace{(\Sigma \cup \{\varepsilon\})}_{\text{input symbol}} \times \underbrace{(\Gamma \cup \{\varepsilon\})}_{\text{pop symbol}} \to \text{subsets of} \left\{ \underbrace{Q}_{\text{state}} \times \underbrace{(\Gamma \cup \{\varepsilon\})}_{\text{push symbol}} \right\}$$

The language of PDA

A PDA is nondeterministic

multiple possible transitions on same input/pop symbol allowed



Transitions may but do not have to push or pop

The language of a PDA is the set of all strings in Σ^*
that can lead the PDA to an accepting state

Example 1

$$L = \{w\#w^R \mid w \in \{0, 1\}^*\}$$

$\#$, $0\#0$, $01\#10$ in L

ε, $01\#1$, $0\#\#0$ not in L

$$\Sigma = \{0, 1, \#\}$$

$$\Gamma = \{0, 1, \$\}$$

Example 1

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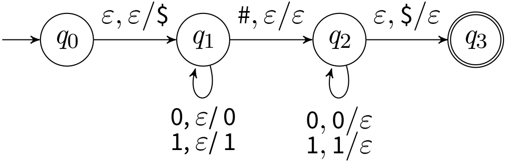

write w on stack

read w from stack

Example 2

$$L = \{ww^R \mid w \in \Sigma^*\}$$

$\Sigma = \{0, 1\}$

ε, 00, 0110 in L

011, 010 not in L

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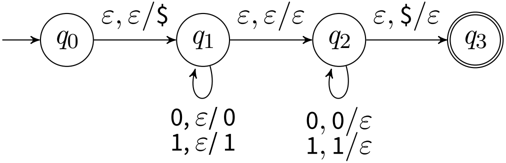
ε, 00, 0110 in L
011, 010 not in L



guess middle of string

Example 3

$$L = \{w \in \Sigma^* \mid w = w^R\}$$

$\Sigma = \{0, 1\}$

ε, 00, 010, 0110 in L

011 not in L

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$$L = \{w \in \Sigma^* \mid w = w^R\}$$

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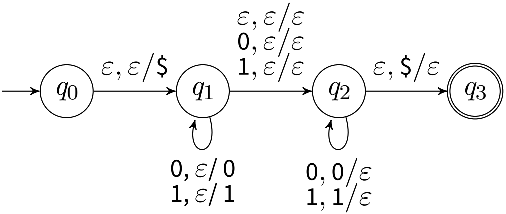

middle symbol can be ε, 0, or 1

$\underbrace{0010}_{x}\,\underbrace{0100}_{x^R}$ or $\underbrace{0010}_{x}\,1\,\underbrace{0100}_{x^R}$

Example 4

$$L = \{0^n 1^m 0^m 1^n \mid n \geqslant 0, m \geqslant 0\}$$

$$\Sigma = \{0, 1\}$$

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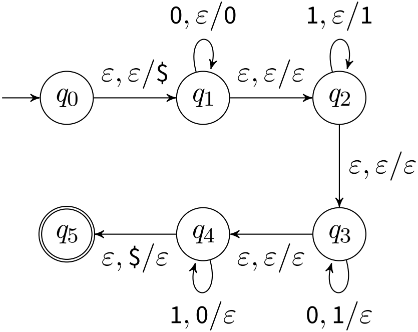

input: $0^n1^m0^m1^n$

stack: 0^n1^m

Example 5

$L =$ same number of 0s and 1s

$\Sigma = \{0, 1\}$

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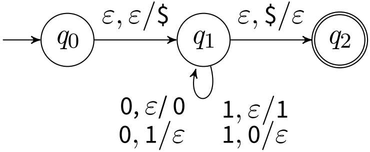
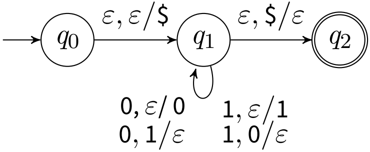
Keep track of excess of 0s or 1s

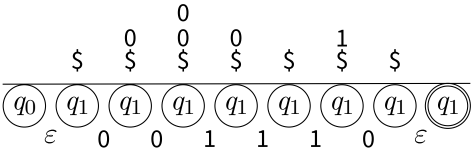
If at the end, the stack is empty, number is equal



Example 5



Example input: 001110

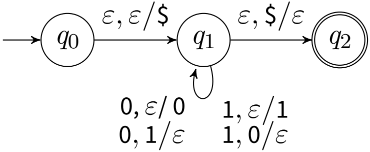


Why does the PDA work?

Example 5

$L =$ same number of 0s and 1s

$\Sigma = \{0, 1\}$



Invariant: In every execution path,
$\#1 - \#0$ on stack = actual $\#1 - \#0$ so far

If $w \notin L$, it must be rejected

Property: In some execution path,
stack consists only of 0s or 1s (or is empty)

If $w \in L$, some execution path will accept