DFA Minimization, Pumping Lemma

CSCI 3130 Formal Languages and Automata Theory

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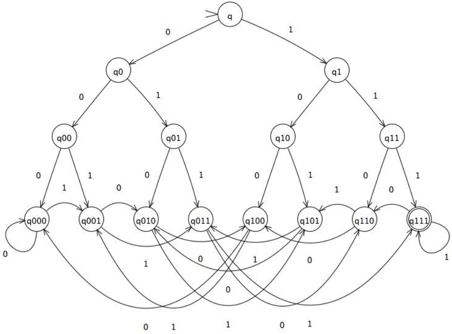
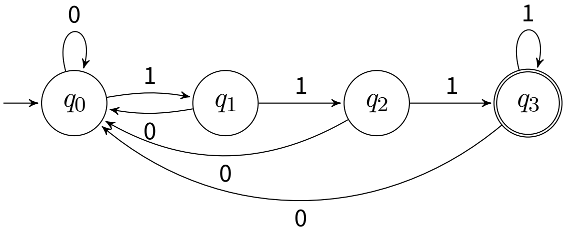
$L =$ strings ending in 111



There is a simpler one…

$L =$ strings ending in 111



Can we do it in 3 states?

Even smaller DFA?

$L =$ strings ending in 111

Intuitively, needs to remember number of ones recently read

We will show

ε, 1, 11, 111 are pairwise distinguishable by L

In other words
$(\varepsilon, 1), (\varepsilon, 11), (\varepsilon, 111), (1, 11), (1, 111), (11, 111)$
are all distinguishable by L

Then use this result from last lecture:

If strings $x_1, \ldots, x_n$ are pairwise distinguishable by L, any DFA accepting L must have at least n states

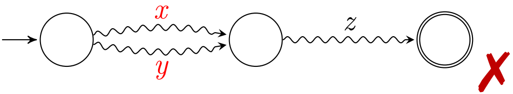
Recap: distinguishable strings

What do we mean by "1 and 11 are distinguishable"?

(x, y) are distinguishable by L if there is string z such that
$xz \in L$ and $yz \notin L$ (or the other way round)

We saw from last lecture

If x and y are distinguishable by L, any DFA accepting L must reach different states upon reading x and y



Distinguishable strings

Why are 1 and 11 distinguishable by L?

$L =$ strings ending in 111

Distinguishable strings

Why are 1 and 11 distinguishable by L?

$L =$ strings ending in 111

Take $z = 1$

$11 \notin L \qquad 111 \in L$

More generally, why are 1^i and 1^j distinguishable by L?

$(0 \leqslant i < j \leqslant 3)$

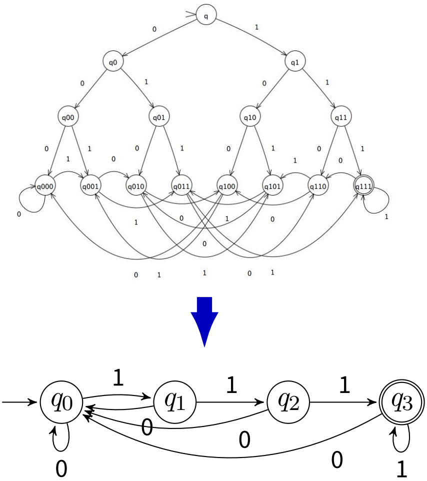
Take $z = 1^{3-j}$

$1^i 1^{3-j} \notin L \qquad 1^j 1^{3-j} \in L$

ε, 1, 11, 111 are pairwise distinguishable by L

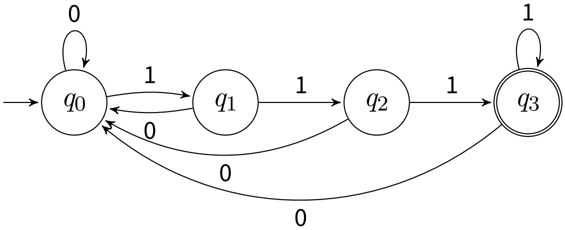
Thus our 4-state DFA is minimal

DFA minimization



We now show how to turn any DFA for L into the minimal DFA for L

Minimal DFA and distinguishability

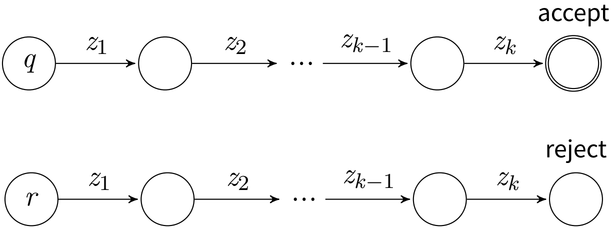


Distinguishable strings must be in different states
Indistinguishable strings may end up in the same state

DFA minimial $\Leftrightarrow$ Every pair of states is distinguishable

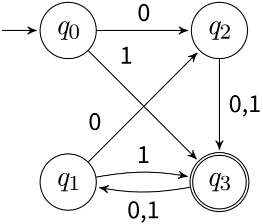
Distinguishable states

Two states q and r are distinguishable if



on the same continuation string $z = z_1 \dots z_k$,
one accepts, but the other rejects

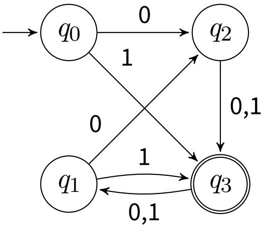
Examples of distinguishable states



Which of the following pairs are distinguishable? by which string?

(q_0, q_3)
(q_1, q_3)
(q_2, q_3)
(q_1, q_2)
(q_0, q_2)
(q_0, q_1)

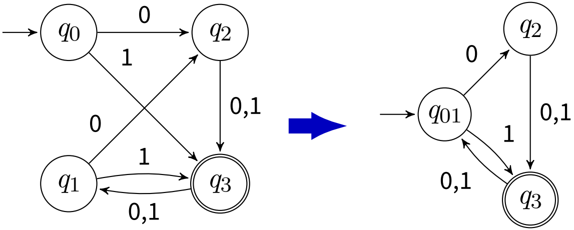
Examples of distinguishable states



Which of the following pairs are distinguishable? by which string?

(q_0, q_3) distinguishable by ε
(q_1, q_3) distinguishable by ε
(q_2, q_3) distinguishable by ε
(q_1, q_2) distinguishable by 0
(q_0, q_2) distinguishable by 0
(q_0, q_1) indistinguishable

Examples of distinguishable states



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(q_0, q_2) distinguishable by 0
(q_0, q_1) indistinguishable

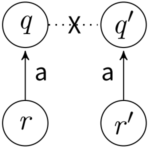
indistinguishable pairs
can be merged

Finding (in)distinguishable states

Phase 1:

If q is accepting and q' is rejecting
Mark (q, q') as distinguishable (X)

Phase 2:



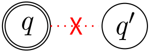
If (q, q') are marked
Mark (r, r') as distinguishable (X)

Phase 3:

Unmarked pairs are indistinguishable
Merge them into groups

Finding (in)distinguishable states

Phase 1:



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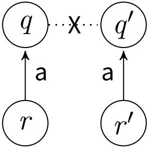

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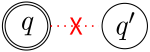

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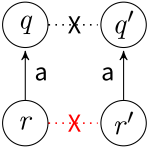
Phase 2:



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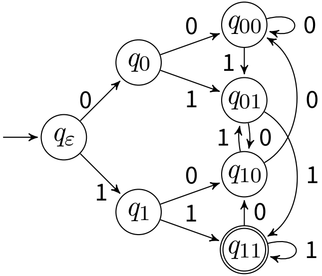
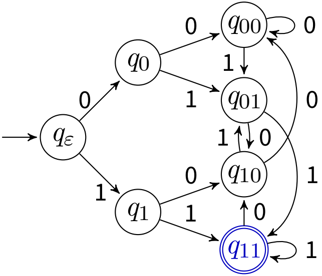
DFA minimization: example



q_0						
q_1						
q_{00}						
q_{01}						
q_{10}						
q_{11}						
	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}

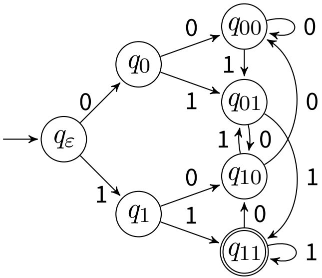
DFA minimization: example



	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}
q_0						
q_1						
q_{00}						
q_{01}						
q_{10}						
q_{11}	X	X	X	X	X	X

(Phase 1) q_{11} is distinguishable from all other states

DFA minimization: example

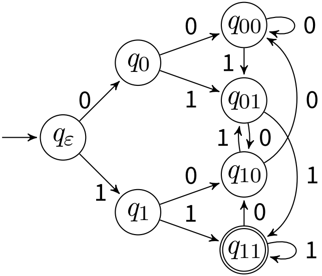


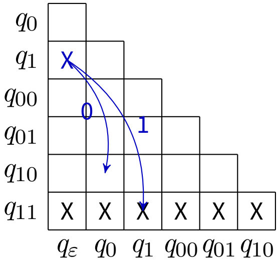
	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}
q_0						
q_1						
q_{00}						
q_{01}						
q_{10}						
q_{11}	X	X	X	X	X	X

(Phase 2) Looking at $(r, r') = (q_\varepsilon, q_0)$

Neither (q_0, q_{00}) nor (q_1, q_{01}) are distinguishable

DFA minimization: example

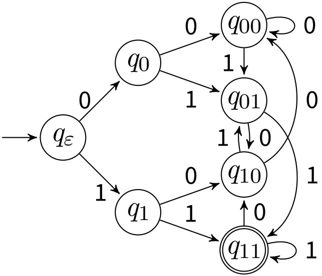




(Phase 2) Looking at $(r, r') = (q_\varepsilon, q_1)$

(q_1, q_{11}) is distinguishable

DFA minimization: example

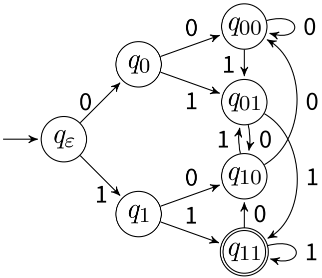


	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}
q_0						
q_1	X	X				
q_{00}			X			
q_{01}	X	X		X		
q_{10}			X		X	
q_{11}	X	X	X	X	X	X

(Phase 2) After going through the whole table once

Now we make another pass

DFA minimization: example

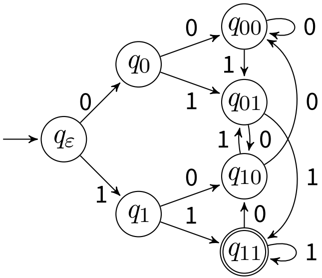


	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}
q_0						
q_1	X	X				
q_{00}			X			
q_{01}	X	X		X		
q_{10}			X		X	
q_{11}	X	X	X	X	X	X

(Phase 2) Looking at $(r, r') = (q_\varepsilon, q_0)$

Neither (q_0, q_{00}) nor (q_1, q_{01}) are distinguishable

DFA minimization: example

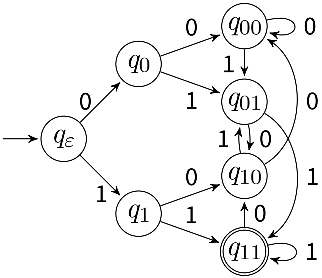


	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}
q_0						
q_1	X	X				
q_{00}			X			
q_{01}	X	X		X		
q_{10}			X		X	
q_{11}	X	X	X	X	X	X

(Phase 2) Looking at $(r, r') = (q_\varepsilon, q_{00})$

Neither (q_0, q_{00}) nor (q_1, q_{01}) are distinguishable

DFA minimization: example

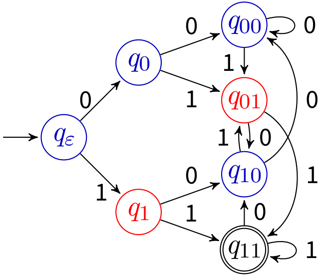


	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}
q_0						
q_1	X	X				
q_{00}			X			
q_{01}	X	X		X		
q_{10}			X		X	
q_{11}	X	X	X	X	X	X

(Phase 2) Nothing changes in the second pass

Ready to go to Phase 3

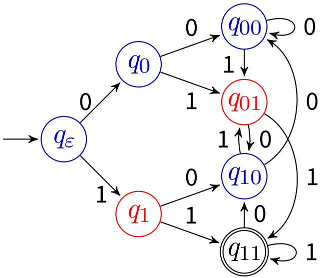
DFA minimization: example



q_0	A					
q_1	X	X				
q_{00}	A	A	X			
q_{01}	X	X	B	X		
q_{10}	A	A	X	A	X	
q_{11}	X	X	X	X	X	X
	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}

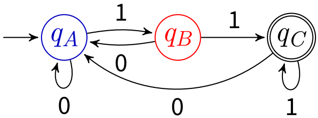
(Phase 3) Merge states into groups (also called equivalence classes)

DFA minimization: example



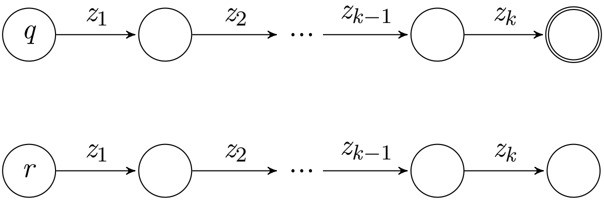
	q_ε	q_0	q_1	q_{00}	q_{01}	q_{10}
q_0	A					
q_1	X	X				
q_{00}	A	A	X			
q_{01}	X	X	B	X		
q_{10}	A	A	X	A	X	
q_{11}	X	X	X	X	X	X

Minimized DFA:



Why it works

Why have we found all distinguishable pairs?



Because we work backwards

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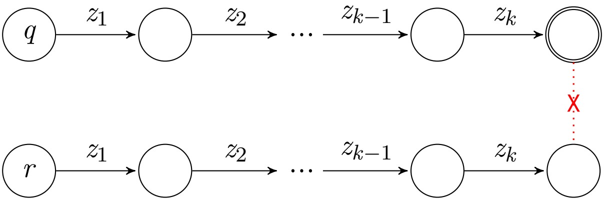

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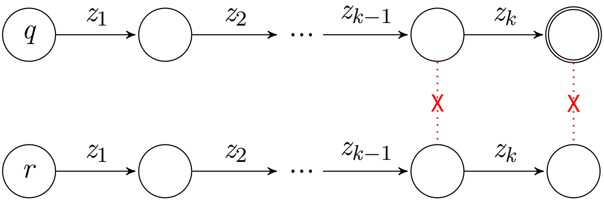

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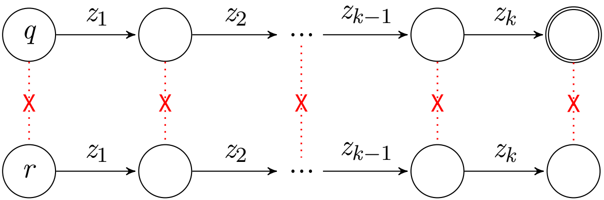


Because we work backwards

Pumping Lemma

Pumping lemma

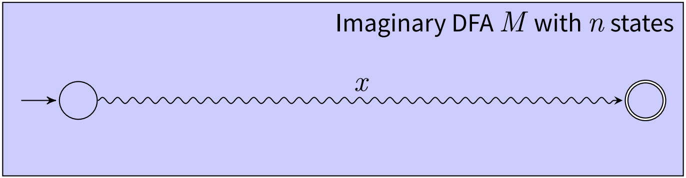
Another way to show some language is irregular

Example

$L = \{0^n 1^n \mid n \geqslant 0\}$ is irregular

We reason by contradiction:
Suppose we have a DFA M for L
Something must be wrong with this DFA
M must accept some strings outside L

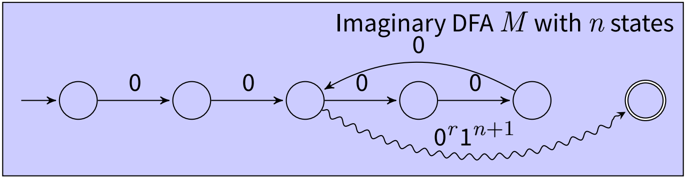
Towards a contradiction



What happens when M gets input $x = 0^{n+1}1^{n+1}$?

M accepts x because $x \in L$

Towards a contradiction



What happens when M gets input $x = 0^{n+1}1^{n+1}$?

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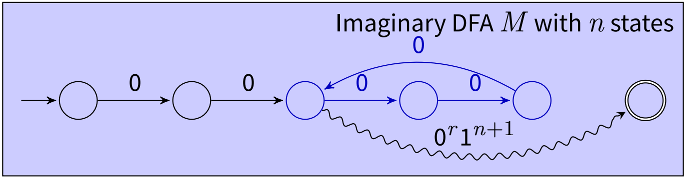
Since M has n states, it must revisit one of its states while reading 0^{n+1}

Towards a contradiction



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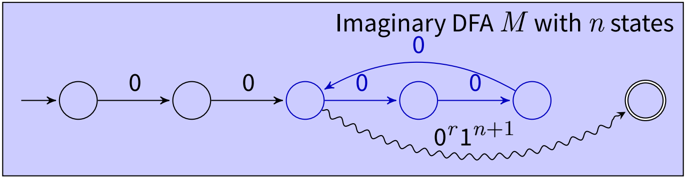
The DFA must contain a cycle with 0s

Towards a contradiction



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M accepts x because $x \in L$

Since M has n states, it must revisit one of its states while reading 0^{n+1}

The DFA must contain a cycle with 0s

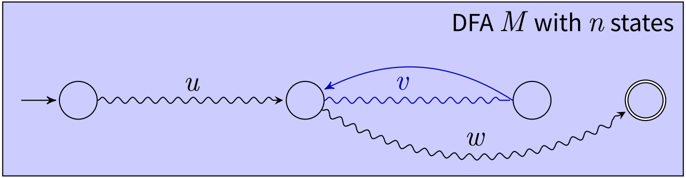
The DFA will also accept strings that go around the cycle multiple times

But such strings have more 0s than 1s and cannot be in L

Pumping lemma for regular languages

For every regular language L, there exists a number n such that
for every string $s \in L$ longer than n symbols,
we can write $s = uvw$ where

1. $|uv| \leqslant n$
2. $|v| \geqslant 1$
3. For every $i \geqslant 0$, the string uv^iw is in L



Proving languages are irregular

For every regular language L, there exists a number n such that
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we can write $s = uvw$ where

1. $|uv| \leqslant n$
2. $|v| \geqslant 1$
3. For every $i \geqslant 0$, the string uv^iw is in L

To show that a language L is irregular
we need to find arbitrarily long s
so that no matter how the lemma splits s into u, v, w (subject to
$|uv| \leqslant n$ and $|v| \geqslant 1$)
we can find $i \geqslant 0$ such that $uv^iw \notin L$

Example

$$L_2 = \{0^m 1^n \mid m > n \geqslant 0\}$$

1. For any n (number of states of an imaginary DFA accepting L_2)
2. There is a string $s = 0^{n+1}1^n$
3. Pumping lemma splits s into uvw $\quad (|uv| \leqslant n$ and $|v| \geqslant 1)$
4. Choose $i = 0$ so that $uv^i w \notin L_2$

Example: 00000011111