Nondeterministic Polynomial Time

CSCI 3130 Formal Languages and Automata Theory

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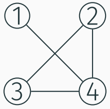
Chinese University of Hong Kong



Graph G

A clique is a subset of vertices that are pairwise adjacent

$\{1, 4\}, \{2, 3, 4\}, \{1\}$ are cliques

An independent set is a subset of vertices that are pairwise non-adjacent

$\{1, 2\}, \{1, 3\}, \{4\}$ are independent sets

A vertex cover is a set of vertices that touches (covers) all edges

$\{2, 4\}, \{3, 4\}, \{1, 2, 3\}$ are vertex covers

These problems

$$\begin{aligned}
\text{CLIQUE} &= \{\langle G, k\rangle \mid \text{Graph } G \text{ has a clique of } k \text{ vertices}\} \\
\textsc{Independent-Set} &= \{\langle G, k\rangle \mid \text{Graph } G \text{ has} \\
&\qquad \text{an independent set of } k \text{ vertices}\} \\
\textsc{Vertex-Cover} &= \{\langle G, k\rangle \mid \text{Graph } G \text{ has} \\
&\qquad \text{a vertex cover of } k \text{ vertices}\}
\end{aligned}$$

What do these problems have in common?

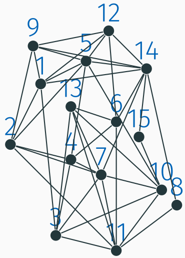
1. Given a candidate solution, we can quickly check if it is valid
2. We don't know how to solve these problems quickly

Checking solutions quickly

If someone told us a candidate solution, we can quickly verify if it is valid

Example: Is $\langle G, 5 \rangle \in$ CLIQUE?

Candidate solution: $\{1, 5, 9, 12, 14\}$



The class NP

A verifier for L is a Turing machine V such that

$$x \in L \iff V \text{ accepts } \langle x, s \rangle \text{ for some } s$$

s is a candidate solution for x

We say V runs in polynomial time if on every input x, it runs in time polynomial in $|x|$ (for every s)

NP is the class of all languages that have polynomial-time verifiers

Example

$V =$ On input $\langle G, k, C \rangle$, where C is a set of vertices in G

If C has size k and all edges between vertices C are present in G

accept

Otherwise reject

Running time: $O(k^2)$

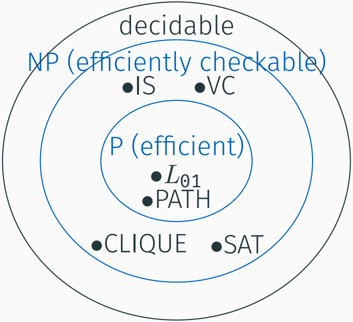
Therefore CLIQUE is in NP

P versus NP

P is contained in NP

because the verifier can ignore the candidate solution

Intuitively, finding solutions can only be harder than verifying them



IS = Independent-Set

VC = Vertex-Cover

We will talk about SAT in the next lecture

Millennium prize problems

In 2000, Clay Math Institute gave 7 problems for the 21st century

- P versus NP (computer science)
- Hodge conjecture
- Poincaré conjecture (Perelman 2006)
- Riemann hypothesis (Hilbert's 8th problem)
- Yang–Mills existence and mass gap
- Navier–Stokes existance and smoothness
- Birth and Swinnerton-Dyer conjecture

P versus NP

Is P equal to NP?

We don't know. But one reason to believe P ≠ NP is that intuitively, searching for a solution is harder than verifying its correctness

For example, solving homework problems (searching for solutions) is harder than grading (verifying the candidate solution is correct)

Searching versus verifying

Mathematician:
Given a mathematical claim, come up with a proof for it

Scientist:
Given a collection of data on some phenomenon, find a theory explaining it

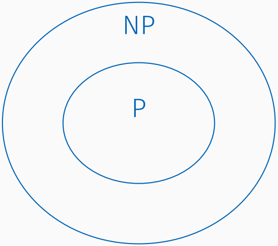
Engineer:
Given a set of constraints (on cost, physical laws, etc), come up with a design (of an engine, bridge, etc) which meets them

Detective:
Given the crime scene, find “who’s done it”

P and NP

P = languages that can be decided on TM in polynomial time (admit efficient algorithms)

NP = languages whose solutions can be verified on a TM in polynomial time (solutions can be checked efficiently)



We believe P $\neq$ NP, but we are not sure

Evidence that NP is bigger than P

$$\begin{aligned}\text{CLIQUE} &= \{\langle G, k\rangle \mid G \text{ is a graph having a clique of } k \text{ vertices}\}\\ \text{IS} &= \{\langle G, k\rangle \mid G \text{ is a graph having}\\ &\qquad \text{an independent set of } k \text{ vertices}\}\\ \text{VC} &= \{\langle G, k\rangle \mid G \text{ is a graph having}\\ &\qquad \text{a vertex cover of } k \text{ vertices}\}\end{aligned}$$

What do they have in common?

- These (and many others) are in NP
- No efficient algorithms are known for solving any of them

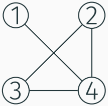
Naive algorithm for solving CLIQUE

$$\text{CLIQUE} = \{\langle G, k \rangle \mid G \text{ is a graph having a clique of } k \text{ vertices}\}$$

Turing machine M : On input $\langle G, k \rangle$

For all subsets S of vertices of size k
  If every pair $u \in S, v \in S, u \neq v$ are adjacent
    accept
  else reject

Example:



Graph G

input:	$\langle G, 3 \rangle$			
subsets:	$\{123\}$	$\{124\}$	$\{134\}$	$\{234\}$
All edges in S?	No	No	No	Yes

Running time analysis

$$\text{CLIQUE} = \{\langle G, k\rangle \mid G \text{ is a graph having a clique of } k \text{ vertices}\}$$

M : On input $\langle G, k\rangle$:

For all subsets S of vertices of size k	$\binom{n}{k}$ subsets
If every pair $u, v \in S$ are adjacent	k^2 pairs
accept	
else reject	
running time:	$k^2\binom{n}{k}$
	$\geqslant 2^n$ when $k = n/2$

Equivalence of certain NP languages

We strongly suspect that problems like CLIQUE, SAT, etc require roughly 2^n time to solve

We do not know how to prove this, but we can prove that

If any one of them can be solved efficiently,
then all of them can be solved efficiently

Next lecture:

All problems such as CLIQUE, SAT, IS, VC are as hard as one another

Moreover, they are at least as hard as any problem in NP