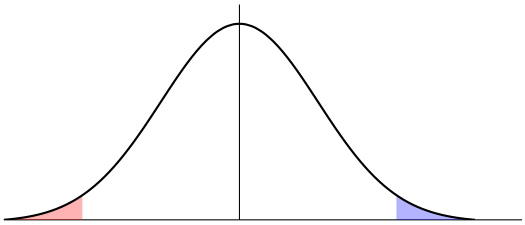
Notes 10: Hypothesis testing

1. Chernoff bounds

Due to Herman Rubin
$X_1, \dots, X_m$ independent $\{0,1\}$-valued random variables
s.t. $\mathbb{P}[X_i = 1] = \mathbb{E}[X_i] = p$ for $1 \leqslant i \leqslant m$
$X \stackrel{\text{def}}{=} X_1 + \cdots + X_m \qquad (\mathbb{E}[X] = mp)$

Theorem 1 (Multiplicative Chernoff)**.** *For all* $0 \leqslant \gamma \leqslant 1$,

$$\mathbb{P}[X \leqslant (1-\gamma)mp] \leqslant e^{-\frac{1}{2}\gamma^2 mp}$$
$$\mathbb{P}[X \geqslant (1+\gamma)mp] \leqslant e^{-\frac{1}{3}\gamma^2 mp}$$



Also true for $X_1, \dots, X_m$ independent $[0,1]$-valued (i.e. bounded) random variables
Many proofs; see e.g. Mulzer *"Five Proofs of Chernoff's Bound with Applications"* if interested
Exponential decay

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2. Hypothesis testing

Fix $h \in \mathcal{H}$, how can we test whether h is bad? (i.e. $\text{err}_{\mathcal{D}}(h,c) = \mathbb{P}_{x\in\mathcal{D}}[h(x) \neq c(x)] \geqslant \varepsilon$)
Solution: Draw m independent labelled samples $(x^1, c(x^1)), \dots, (x^m, c(x^m))$,

Compute (empirical error) $\widehat{\text{err}} \stackrel{\text{def}}{=} \dfrac{\#\text{samples } s.t.\ h(x^i) \neq c(x^i)}{m}$

By Chernoff bound, $\widehat{\text{err}} \approx \text{err}_{\mathcal{D}}(h,c)$
e.g. If h is bad, $p \stackrel{\text{def}}{=} \text{err}_{\mathcal{D}}(h,c) \geqslant \varepsilon$,

$$\mathbb{P}\left[\widehat{\text{err}} \leqslant \frac{\varepsilon}{2}\right] \leqslant e^{-\frac{1}{8}mp} \leqslant e^{-\frac{1}{8}\varepsilon m}$$

Further Improved Algorithm: Similar to Improved Algorithm
But only cover $1 - \varepsilon/2$ fraction of positive samples using $S_{i_1}, \dots, S_{i_k}$
Number of sets needed $k \leqslant \text{OPT} \cdot \ln(2/\varepsilon)$ (why?)
Can show that $O\left(\frac{1}{\varepsilon}(\ln\frac{1}{\delta} + s\ln\frac{1}{\varepsilon}\ln n)\right)$ samples suffices (exercise)