ENGG1410-F Tutorial 9

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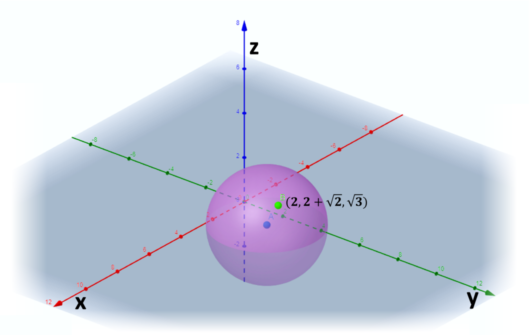
Problem 1.

Consider the sphere $(x-1)^2+(y-2)^2+z^2=6$.

1. Give a normal vector of the sphere at point $(2, 2+\sqrt{2}, \sqrt{3})$.
2. Give the equation of the tangent plane at point $(2, 2+\sqrt{2}, \sqrt{3})$.



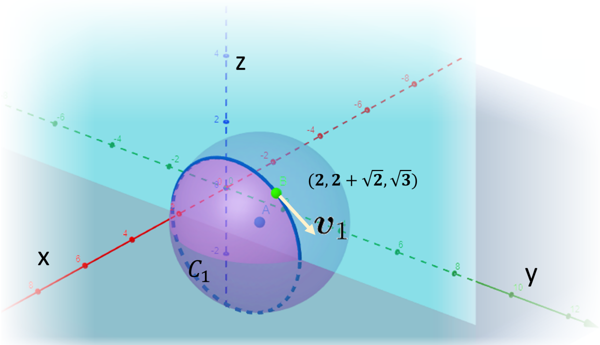
Problem 1 - Solution.

See Problem 1 of "Excercise: Surfaces".

Problem 2.

Consider again the sphere $(x-1)^2 + (y-2)^2 + z^2 = 6$.

1. Let C_1 be the curve on the sphere satisfying $x = 2$. Give a tangent vector $\boldsymbol{v}_1$ of C_1 at point $(2, 2+\sqrt{2}, \sqrt{3})$.
2. Let C_2 be the curve on the sphere satisfying $y = 2+\sqrt{2}$. Give a tangent vector $\boldsymbol{v}_2$ of C_2 at point $(2, 2+\sqrt{2}, \sqrt{3})$.
3. Compute $\boldsymbol{v}_1 \times \boldsymbol{v}_2$.



Problem 2 - Solution.

See Problem 2 of "Excercise: Surfaces".

Problem 3.

Let C be the arc on the curve $\boldsymbol{r}(t) = [(\cos t)^3, (\sin t)^3]$ defined by increasing t from 0 to $\pi/2$. Calculate $\int_C ds$, namely, the length of C.

Problem 3 - Solution.

First, represent C as the vector function $\boldsymbol{r}(t) = [(\cos t)^3, (\sin t)^3]$ with t ranging from 0 to $\pi/2$. Then:

$$\begin{aligned}
\int_C ds &= \int_0^{\pi/2} ds = \int_0^{\pi/2} \frac{ds}{dt} dt \\
&= \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
&= \int_0^{\pi/2} \sqrt{\Big(3(\cos t)^2(-\sin t)\Big)^2 + \Big(3(\sin t)^2(\cos t)\Big)^2} \, dt \\
&= 3\int_0^{\pi/2} \cos t \sin t \, dt = \frac{3}{2} \sin^2 t \Big|_0^{\pi/2} = \frac{3}{2}
\end{aligned}$$

Problem 4.

Let C be the line segment from point $p(1,2,3)$ to point $q(8,7,6)$. Calculate $\int_C (x+z^2)ds$.

Problem 4 - Solution.

See Problem 2 of "Excercise: Line Integrals by Arc Length".

Problem 5.

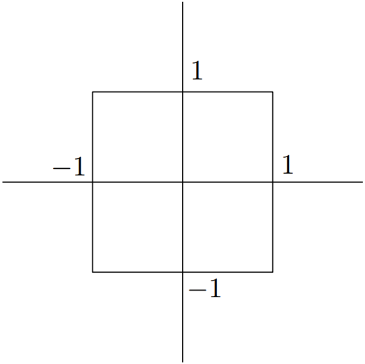
Let C be the circle $x^2 + y^2 = 1$. Calculate $\int_C y ds$.

Problem 5 - Solution.

See Problem 3 of "Excercise: Line Integrals by Arc Length".

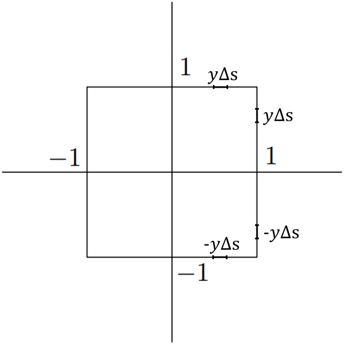
Problem 6.

Let C be the boundary of the square shown below:
Calculate $\int_C y ds$.



Problem 6 - Solution.

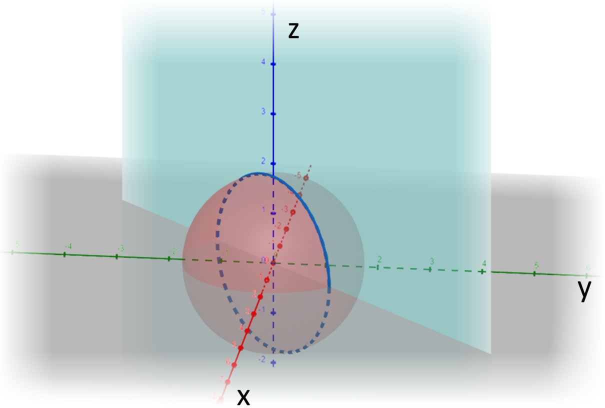
$\int_C y ds = 0$ can be directly inferred from the definition of "line integral by arc length":
Break each edge into subintervals, and argue that each subinterval will be "canceled" by another subinterval mirrored about x-axis.



See also Problem 4 of "Excercise: Line Integrals by Arc Length".

Problem 7.

Let C be the intersection of two surfaces: sphere $x^2 + y^2 + z^2 = 3$ and plane $x = y$. Calculate $\int_C x^2 ds$.



Problem 7 - Solution.

See Problem 5 of "Excercise: Line Integrals by Arc Length".