ENGG1410F Tutorial
Introduction to Page Ranks

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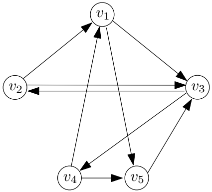
Google made its first debut with page ranks, which represent a technique for ranking the webpages on the Internet by importance. Today we will give a short introduction to this technique. Interestingly, at its core, the technique requires computing just an eigenvector.

Let us model the Internet as a graph. Each webpage is represented as a node. Given two nodes $v_1, v_2 \in V$, there is an edge from v_1 to v_2 if the webpage v_1 has a hyperlink to the webpage v_2.



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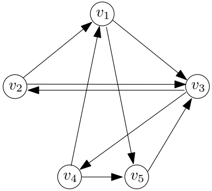
3. With probability $1 - \alpha$:
   3.1 Set u to a random webpage in the Internet.
   3.2 Repeat from Step 2.

The value of α is often set to 0.85 in practice.



For example, suppose we are at v_3. Conceptually this is what we do:

- Toss a coin that heads with probability α.
- If the coin comes up heads, jump to v_2 or v_4 with equal chance.
- If the coin comes up tails, jump to v_1, v_2, ..., v_5 with equal chance.

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But this is a wrong question, because the process is random, such that there won't be a deterministic answer. The correct question to ask is:

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The probability is the page rank of v_1. Of course, the same question can also be asked about any other page.

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Why the number 100000000000? Interestingly, the theory of random walks (which we will not get into today) tells us that the probability remains the same as long as a sufficiently large number of steps have been performed! In other words, it won't matter if you replace 100000000000 with, say, 100000000001!

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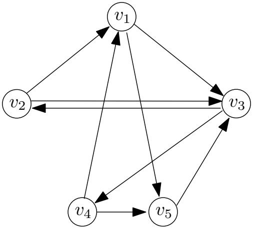
The rationale behind page ranks is this:

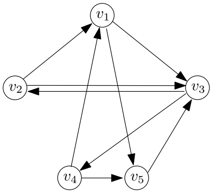
A page v is more "important", i.e., having a higher page rank, if a random surfer has a larger chance landing on v after a sufficiently large number of steps.



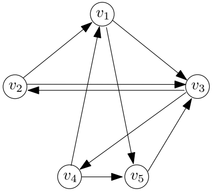
The page ranks of $v_1, \ldots, v_5$ are 0.1716, 0.1666, 0.3214, 0.1666, and 0.1737, respectively. Note that the sum of all the page ranks is 1.

Remaining question: How to calculate them?

Let n be the number of nodes.
Define $\boldsymbol{M} = [m_{ij}]$ as an $n \times n$ matrix
where m_{ij} is the probability of moving from node v_j to node v_i.



For example, when $\alpha = 0.85$, $m_{23} = 0.455$. Why? See next.



Recall: Suppose we are at v_3. Conceptually this is what we do:

- Toss a coin that heads with probability α.
- If the coin comes up heads, jump to v_2 or v_4 with equal chance.
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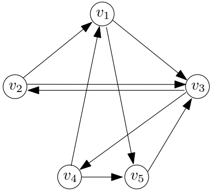
So the probability to go from v_3 to v_2 is:

$$\alpha/2 + (1 - \alpha)/5$$

which is 0.455 for $\alpha = 0.85$.



You can verify:

$$\boldsymbol{M} = \begin{bmatrix} 0.03 & 0.455 & 0.03 & 0.455 & 0.03 \\ 0.03 & 0.03 & 0.455 & 0.03 & 0.03 \\ 0.455 & 0.455 & 0.03 & 0.03 & 0.88 \\ 0.03 & 0.03 & 0.455 & 0.03 & 0.03 \\ 0.455 & 0.03 & 0.03 & 0.455 & 0.03 \end{bmatrix}$$

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- $\boldsymbol{M}$ must have an eigenvalue 1.
- The page ranks make an eigenvector under the eigenvalue 1!

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In our example:

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You can verify that $\begin{bmatrix} 0.1716 \\ 0.1666 \\ 0.3214 \\ 0.1666 \\ 0.1737 \end{bmatrix}$ is indeed an eigenvector of $\boldsymbol{M}$ under eigenvalue 1.

We now have an algorithm to compute the page ranks:

1. Obtain $\boldsymbol{M}$.
2. Obtain an arbitrary eigenvector $\boldsymbol{p}$ of $\boldsymbol{M}$ under the eigenvalue 1.
3. Scale $\boldsymbol{p}$ into $c\boldsymbol{p}$ with a proper real number c so that all components of $c\boldsymbol{p}$ add up to 1.
4. $c\boldsymbol{p}$ now stores the page ranks of all vertices.