ENGG1410-F: Quiz 2

Name: **Student ID:**

Write all your answers on this sheet, and use the back if necessary.

Problem 1 (20%). Compute the inverse of the following matrix

$$\begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Solution.

$$\begin{bmatrix} 0 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 1 & -1/2 & 1 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 0 & -1/2 & 1 & -1 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1/4 & 1/2 & -1/2 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Hence, the answer is $\begin{bmatrix} -1/4 & 1/2 & -1/2 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Problem 2 (20%). Find the dimension of the following set of vectors:

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x^2 + y^2 + z^2 = 1 \right\}.$$

You need to show the details of your work.

Solution. Clearly, vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are all in the set. Since these vectors constitute a linearly independent set, we know that the dimension of the set is at least 3. On the other hand, since each vector is 3×1, the dimension of the set is at most 3. It thus follows that the set has dimension exactly 3.

Problem 3 (60%). Diagonalize the following matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

into the form $\boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^{-1}$ where $\boldsymbol{B}$ is a diagonalize matrix. You only need to show the details of $\boldsymbol{Q}$ and $\boldsymbol{B}$ (namely, you do not need to give the details of $\boldsymbol{Q}^{-1}$).

Solution. We first obtain the characteristic equation of $\boldsymbol{A}$:

$$\begin{aligned} |\boldsymbol{A} - \lambda \boldsymbol{I}| &= 0 \Rightarrow \\ \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} &= 0 \Rightarrow \\ \lambda^2 - 1 &= 0. \end{aligned}$$

Hence, the eigenvalues are: $\lambda_1 = 1, \lambda_2 = -1$. Since these eigenvalues are distinct, it suffices to find an arbitrary eigenvector for each eigenvalue.

Next we obtain the eigenspace of λ_1, namely, the set of $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfying:

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow$$

Find an arbitrary non-zero solution, e.g., $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

We then obtain the eigenspace of λ_2, namely, the set of $\begin{bmatrix} x \\ y \end{bmatrix}$ satisfying:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow$$

Find an arbitrary non-zero solution, e.g., $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Therefore, $\boldsymbol{Q} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.