Lecture Notes: Geometry of Vectors

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Given an integer $d \geq 1$, we use $\mathbb{R}^d$ to denote the d-dimensional space where each dimension has a domain of $\mathbb{R}$ (recall that $\mathbb{R}$ is the set of real values).

Recall that we have defined a *vector* as either a $d \times 1$ matrix (column vector) or a $1 \times d$ matrix (row vector). Our discussion henceforth will by default refer to row vectors simply as "vectors" (but the discussion can be generalized to column vectors in an obvious manner). Henceforth, a *d-dimensional vector* has the form $[v_1, v_2, ..., v_d]$, where each component v_i $(1 \leq i \leq d)$ is a real value. Boldfaces will be used to denote vectors, e.g., $\boldsymbol{v} = [v_1, v_2, ..., v_d]$. We use $\mathbf{0}$ to represent the specific vector $[0, 0, ..., 0]$ called the *zero vector*. Recall that the *length*, also called the *norm*, of a vector $\boldsymbol{v} = [v_1, v_2, ..., v_d]$ is defined to be

$$|\boldsymbol{v}| = \sqrt{\sum_{i=1}^{d} v_i^2}.$$

We refer to $\boldsymbol{v}$ as a *unit vector* if $|\boldsymbol{v}| = 1$.

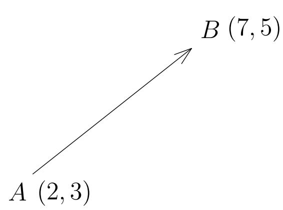
Let $p_1 = (a_1, a_2, ..., a_d)$ and $p_2 = (b_1, b_2, ..., b_d)$ be two points in $\mathbb{R}^d$. They define a *directed segment* $\overrightarrow{p_1p_2}$ which is the segment connecting p_1 and p_2, but also carrying a direction from p_1 to p_2. As shown below, every directed segment defines a vector:

Definition 1. *Given a directed segment $\overrightarrow{p_1p_2}$ where the points $p_1 = (a_1, a_2, ..., a_d), p_2 = (b_1, b_2, ..., b_d)$, we say that it* **defines** *a vector $[v_1, ..., v_d]$ where*

$$v_i \quad = \quad b_i - a_i$$

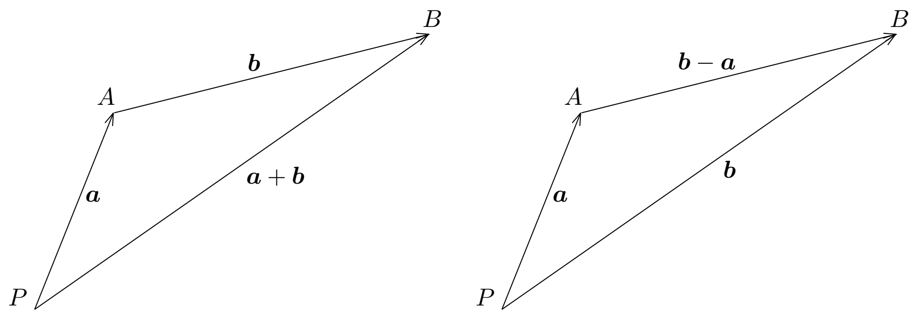
for all $i \in [1, d]$.

For example, consider the segment $\overrightarrow{AB}$ shown below. They define the vector $[5, 2]$. Note that the length of this vector is precisely the length of $\overrightarrow{AB}$. For convenience, we will simply use the term "vector $\overrightarrow{p_1p_2}$" to refer to the vector it defines. For example, $[5, 2]$ is the vector $\overrightarrow{AB}$.



The above geometry offers an intuitive understanding about vector additions and subtractions, as shown next:

Lemma 1. *Suppose that $\overrightarrow{PA}$ and $\overrightarrow{AB}$ define vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. Then, $\overrightarrow{PB}$ defines vector $\boldsymbol{a} + \boldsymbol{b}$; see Figure 1a.*



(a) (b)

Figure 1: Geometric view of vector addition and subtraction

Proof. Suppose that $\boldsymbol{a} = [a_1, a_2, ..., a_d]$ and $\boldsymbol{b} = [b_1, b_2, ..., b_d]$. Also, assume that $P = (p_1, p_2, ..., p_d)$, $A = (x_1, x_2, ..., x_d)$, and $B = (y_1, y_2, ..., y_d)$.

Because $\overrightarrow{PA}$ and $\overrightarrow{AB}$ define $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, we know

$$\begin{aligned} a_i &= x_i - p_i, \forall i \in [1, d] \\ b_i &= y_i - x_i, \forall i \in [1, d]. \end{aligned}$$

It thus follows that

$$a_i + b_i = y_i - p_i, \forall i \in [1, d].$$

Therefore, $\overrightarrow{PB}$ defines $\boldsymbol{a} + \boldsymbol{b}$. □

Corollary 1. *Suppose that* $\overrightarrow{PA}$ *and* $\overrightarrow{PB}$ *define* $\boldsymbol{a}$ *and* $\boldsymbol{b}$*, respectively. Then,* $\overrightarrow{AB}$ *defines* $\boldsymbol{b} - \boldsymbol{a}$*; see Figure 1b.*

Finally, when $d = 3$, we define 3 special unit vectors:

$$\boldsymbol{i} = [1, 0, 0],\ \boldsymbol{j} = [0, 1, 0],\ \boldsymbol{k} = [0, 0, 1].$$

This allows us to represent a 3d vector $\boldsymbol{v} = [v_1, v_2, v_3]$ as $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$ (note that all the operators in this equation are now well defined). Similarly, when $d = 2$, we define 2 special unit vectors:

$$\boldsymbol{i} = [1, 0],\ \boldsymbol{j} = [0, 1].$$

A 2d vector $\boldsymbol{v} = [v_1, v_2]$ can therefore be represented as $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j}$.