# Lecture Notes: Solutions of a Linear System

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We can now utilize the materials we have learned to strengthen our understanding about linear systems. Recall that a linear system on $n$ variables is a set of $m$ equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ &... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m \end{aligned}$$

As before, introduce:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... \\ a_{n1} & a_{n2} & ... & a_{mn} \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{bmatrix}, \text{ and } \boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_m \end{bmatrix}$$

The linear system can be represented as:

$$\boldsymbol{Ax} = \boldsymbol{b}. \tag{1}$$

Henceforth, we will refer to $\boldsymbol{A}$ as the *coefficient matrix*. The *augmented matrix* of $\boldsymbol{A}$ is:

$$\tilde{\boldsymbol{A}} = \left[\begin{array}{cccc|c} a_{11} & a_{12} & ... & a_{1n} & b_1 \\ a_{21} & a_{22} & ... & a_{2n} & b_2 \\ ... \\ a_{m1} & a_{m2} & ... & a_{mn} & b_n \end{array}\right]$$

Recall that if a linear system has at least one solution, we say that the system is *consistent*; otherwise, it is *inconsistent*. The next theorem characterizes the conditions for the system to be consistent.

**Theorem 1 (Consistency Criterion).** *Linear system* (1) *has:*

1. *no solution if and only if rank* $\boldsymbol{A} <$ *rank* $\tilde{\boldsymbol{A}}$*;*
2. *exactly one solution if and only if rank* $\boldsymbol{A} =$ *rank* $\tilde{\boldsymbol{A}} = n$*;*
3. *infinitely many solutions if and only if rank* $\boldsymbol{A} =$ *rank* $\tilde{\boldsymbol{A}} < n$*.*

*Proof.* The theorem follows directly from our earlier discussion on Gauss elimination and rank calculation. □

**Example 1.** Consider the following linear system:

$$\begin{aligned} 3x_2 &= 4 \\ 2x_1 + x_2 + 6x_3 &= 3 \\ 4x_1 + 5x_2 + 12x_3 &= 10. \end{aligned}$$

The coefficient matrix $\boldsymbol{A}$ and the augmented matrix $\tilde{\boldsymbol{A}}$ are

$$\boldsymbol{A} = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 1 & 6 \\ 4 & 5 & 12 \end{bmatrix}, \tilde{\boldsymbol{A}} = \begin{bmatrix} 0 & 3 & 0 & 4 \\ 2 & 1 & 6 & 3 \\ 4 & 5 & 12 & 10 \end{bmatrix}$$

which can be converted to the following matrices of row echelon form respectively:

$$\begin{aligned} \boldsymbol{A} &\Rightarrow \begin{bmatrix} 2 & 1 & 6 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \tilde{\boldsymbol{A}} &\Rightarrow \begin{bmatrix} 2 & 1 & 6 & 3 \\ 0 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

We thus know that $rank\ \boldsymbol{A} = rank\ \tilde{\boldsymbol{A}} < 3$. Hence, the system has infinitely many solutions. □

**Corollary 1.** *Suppose that $\boldsymbol{A}$ is an $n \times n$ matrix, i.e., the linear system* (1) *has $n$ equations on $n$ variables. Then, the linear system has a unique solution if and only if $det(\boldsymbol{A}) \neq 0$.*

*Proof.* *If Direction.* From $det(\boldsymbol{A}) \neq 0$ we know that $rank\ \boldsymbol{A} = n$. This means that $rank\ \tilde{\boldsymbol{A}}$ must also be $n$ because $\tilde{\boldsymbol{A}}$ has only $n$ rows. Hence, Theorem 1 shows that (1) has a unique solution.

*Only-If Direction.* When (1) has a unique solution, by Theorem 1, we know that $rank\ \boldsymbol{A} = n$. Therefore, $det(\boldsymbol{A}) \neq 0$. □

We state the next result without proof:

**Theorem 2 (Cramer's Rule).** *Consider the linear system in* (1) *with $\boldsymbol{A}$ being an $n \times n$ matrix. When $det(\boldsymbol{A}) \neq 0$, the system has a unique solution:*

$$\begin{aligned} x_1 &= \frac{det(\boldsymbol{A}_1)}{det(\boldsymbol{A})} \\ x_2 &= \frac{det(\boldsymbol{A}_2)}{det(\boldsymbol{A})} \\ &\cdots \\ x_n &= \frac{det(\boldsymbol{A}_n)}{det(\boldsymbol{A})} \end{aligned}$$

*where $\boldsymbol{A}_i$ ($1 \leq i \leq n$) is the matrix obtained by replacing the $i$-th column of $\boldsymbol{A}$ with $\boldsymbol{b}$.*

**Example 2.** Consider the system:

$$\begin{aligned} 2x_1 + x_2 &= 3 \\ x_1 + 2x_2 &= 1 \end{aligned}$$

The coefficient matrix equals

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Since $det(\boldsymbol{A}) \neq 0$, the system has a unique solution. Define:

$$\boldsymbol{A}_1 = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \boldsymbol{A}_2 = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

Thus, by Theorem 2, we have:

$$\begin{aligned} x_1 &= \frac{det(\boldsymbol{A}_1)}{det(\boldsymbol{A})} = \frac{5}{3} \\ x_2 &= \frac{det(\boldsymbol{A}_2)}{det(\boldsymbol{A})} = \frac{-1}{3}. \end{aligned}$$

□