Lecture Notes: Matrix Inverse

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1 Inverse Definition

We use $\boldsymbol{I}$ to represent *identity matrices*, namely, diagonal matrices where all the elements on the main diagonal are 1.

Definition 1. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ matrices. If $\boldsymbol{AB} = \boldsymbol{I}$, then we say that $\boldsymbol{B}$ is the* **inverse** *of $\boldsymbol{A}$, denoted as $\boldsymbol{A}^{-1}$.*

For example, let

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 4 \\ 0 & -2 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} -4 & 1 & -4 \\ 1/2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

You can verify that $\boldsymbol{AB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Hence, $\boldsymbol{B} = \boldsymbol{A}^{-1}$. Some square matrices have no inverses. For example, $\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 4 \\ 2 & 4 & 8 \end{bmatrix}$ has no inverse (you are encouraged to make an attempt to find it, and see where you will get stuck).

Definition 2. *An $n \times n$ matrix $\boldsymbol{A}$ is said to be*

- singular *if it does not have an inverse;*
- non-singular *if it does.*

2 Inverse Existence and Uniqueness

Lemma 1. *An $n \times n$ matrix $\boldsymbol{A}$ has an inverse if and only if $\boldsymbol{A}$ has rank n (equivalently, $det(\boldsymbol{A}) \neq 0$).*

Proof. <u>*If-Direction.*</u> If $\boldsymbol{A}$ has rank n, the linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has a unique solution for any $\boldsymbol{b}$. Denote by $\boldsymbol{b}_i$ the i-th column ($1 \le i \le n$) of the $n \times n$ identity matrix $\boldsymbol{I}$, and $\boldsymbol{x}_i$ the solution of the system $\boldsymbol{Ax}_i = \boldsymbol{b}_i$. Then, we obtain $\boldsymbol{A}^{-1}$ by placing $\boldsymbol{x}_i$ as the i-th column of $\boldsymbol{A}^{-1}$, for each $i \in [1, n]$.

<u>*Only-If Direction.*</u> Now consider that $\boldsymbol{A}^{-1}$ exists, i.e., there is a matrix $\boldsymbol{B}$ such that $\boldsymbol{AB} = \boldsymbol{I}$. Hence, $\boldsymbol{B}^T\boldsymbol{A}^T = \boldsymbol{I}$.

Given any linear system $\boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{b}$, we have $\boldsymbol{B}^T\boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{B}^T\boldsymbol{b}$, which gives $\boldsymbol{x} = \boldsymbol{B}^T\boldsymbol{b}$. This indicates that $\boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{b}$ has at least one solution[1]. More subtly, this also implies that $\boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{b}$

[1] Strictly speaking, we still need to show $\boldsymbol{A}^T(\boldsymbol{B}^T\boldsymbol{b})$ equals $\boldsymbol{b}$. This is in fact a simple corollary of Lemma 2 (which we will prove shortly), and is left to you to figure out.

has a unique solution. To see this, suppose that there was another solution $\boldsymbol{x'} \neq \boldsymbol{x}$. By the same derivation, we get $\boldsymbol{x'} = \boldsymbol{B}^T\boldsymbol{b} = \boldsymbol{x}$, giving a contradiction.

$\boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{b}$ having a unique solution means that $\boldsymbol{A}^T$ has rank n. It thus follows that the rank of $\boldsymbol{A}$ is also n. □

Corollary 1. *If $\boldsymbol{A}^{-1}$ exists, it is unique.*

Proof. This in fact follows from the argument we used to prove the "if-direction" of Lemma 1. □

Lemma 2. *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ matrices. If $\boldsymbol{AB} = \boldsymbol{I}$, then $\boldsymbol{BA} = \boldsymbol{I}$.*

Proof. We first prove that $\boldsymbol{B}$ has rank n. Indeed:

$$rank\boldsymbol{B} = rank\boldsymbol{B}^T \geq rank(\boldsymbol{B}^T\boldsymbol{A}^T) = rank(\boldsymbol{AB}) = rank(\boldsymbol{I}) = n.$$

Hence, by Lemma 1, we know that $\boldsymbol{B}$ has an inverse, say, $\boldsymbol{X}$; namely, $\boldsymbol{BX} = \boldsymbol{I}$. Equipped with this, we can show $\boldsymbol{BA} = \boldsymbol{I}$ as follows:

$$\begin{aligned} \boldsymbol{BA} &= \boldsymbol{BAI} \\ &= \boldsymbol{BABX} \\ &= \boldsymbol{B}(\boldsymbol{AB})\boldsymbol{X} \\ &= \boldsymbol{BIX} \\ &= \boldsymbol{BX} \\ &= \boldsymbol{I}. \end{aligned}$$

□

In other words, if $\boldsymbol{B}$ is $\boldsymbol{A}^{-1}$, then $\boldsymbol{A} = \boldsymbol{B}^{-1}$.

3 More Properties of Inverses

Lemma 3. *Let $\boldsymbol{A}, \boldsymbol{B}$ be $n \times n$ non-singular matrices. Then, $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.*

Proof.

$$(\boldsymbol{AB})(\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}) = \boldsymbol{A}(\boldsymbol{BB}^{-1})\boldsymbol{A}^{-1} = \boldsymbol{AIA}^{-1} = \boldsymbol{AA}^{-1} = \boldsymbol{I}.$$

Hence, $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$. □

Lemma 4. *Let $\boldsymbol{A}$ be an $n \times n$ non-singular matrix. Then, $(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$.*

Proof. It suffices to prove that $\boldsymbol{A}^T(\boldsymbol{A}^{-1})^T = \boldsymbol{I}$. This is true because

$$(\boldsymbol{A}^T(\boldsymbol{A}^{-1})^T)^T = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}.$$

□

Lemma 5. *Let $\boldsymbol{A}$ be an $n \times n$ non-singular matrix. Then, $det(\boldsymbol{A}^{-1}) = 1/det(\boldsymbol{A})$.*

Proof.

$$det(\boldsymbol{A}) \cdot det(\boldsymbol{A}^{-1}) = det(\boldsymbol{AA}^{-1}) = det(\boldsymbol{I}) = 1.$$

The lemma thus follows. □

Lemma 6. *Let $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ be $n \times n$ matrices. If $\boldsymbol{A}$ is non-singular and $\boldsymbol{AB} = \boldsymbol{AC}$, then $\boldsymbol{B} = \boldsymbol{C}$.*

Proof. From $\boldsymbol{AB} = \boldsymbol{AC}$, we have $\boldsymbol{A}^{-1}\boldsymbol{AB} = \boldsymbol{A}^{-1}\boldsymbol{AC}$, which gives $\boldsymbol{B} = \boldsymbol{C}$. □

4 Inverse Computation: Gauss-Jordan Elimination

We will use an example to illustrate how to compute the inverse of a matrix $\boldsymbol{A}$. Consider that

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 4 \\ 0 & -2 & 1 \end{bmatrix}$$

Suppose that we do not know what is $\boldsymbol{A}^{-1}$; hence, we assume:

$$\boldsymbol{A}^{-1} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

Remember that we want

$$\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 4 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is essentially to solve three linear systems:

$$\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 4 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

$$\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 4 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 4 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{3}$$

Now we can focus on solving these systems respectively using Gauss Elimination. For example, to solve the linear system (1), we look at the augmented matrix:

$$\left[\begin{array}{ccc|c} 0 & 2 & 0 & 1 \\ 1 & 0 & 4 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & -2 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \tag{4}$$

Usually, we would start back substitution from here, but now we take a different approach. In particular, we will show that (since the system has a unique solution) it is possible to get rid of back substitution, but instead, continue to use elementary row operations to make the left side of the vertical bar an identity matrix. Then, the solution of the system will present itself. Specifically:

$$(4) \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

It is thus clear that $x_{11} = -4, x_{21} = 1/2, x_{31} = 1$. The above method is an extension of Gauss elimination, and is referred to as *Gauss-Jordan elimination*.

Now you may proceed to solve (2) and (3) in the same way. You will then realize that the operations done to the left of the vertical line are *always* the same. Motivated by this, we can solve all three systems (1)-(3) *together* in one go, as illustrated below:

$$\left[\begin{array}{ccc|ccc} 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & \text{-}2 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & \text{-}2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \text{-}4 & 1 & \text{-}4 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

What is now on the right side of the bar is exactly $\boldsymbol{A}^{-1}$. It is important to observe that the above process has in fact embedded the Gauss-Jordan elimination for solving all three linear systems (1)-(3).

5 Inverse Formula

It is possible to give a general formula for the inverse of an $n \times n$ non-singular matrix $\boldsymbol{A}$. As before, given $i, j \in [1, n]$, we denote by $\boldsymbol{M}_{ij}$ the $(n-1) \times (n-1)$ matrix obtained from $\boldsymbol{A}$ after discarding its i-th row and j-th column. Also, define:

$$C_{ij} \quad = \quad (-1)^{i+j} \cdot det(\boldsymbol{M}_{ij}).$$

Then we have:

Lemma 7.

$$\boldsymbol{A}^{-1} \quad = \quad \frac{1}{det(\boldsymbol{A})} \left[\begin{array}{cccc} C_{11} & C_{21} & ... & C_{n1} \\ C_{12} & C_{22} & ... & C_{n2} \\ ... & ... & ... & ... \\ C_{1n} & C_{2n} & ... & C_{nn} \end{array}\right].$$

We skip a proof of the lemma, but illustrate it with an example.

Example 1. Consider once again

$$\boldsymbol{A} = \left[\begin{array}{ccc} 0 & 2 & 0 \\ 1 & 0 & 4 \\ 0 & \text{-}2 & 1 \end{array}\right],$$

We have: $det(\boldsymbol{A}) = -2$. Also:

$$\begin{aligned}
\boldsymbol{M}_{11} &= \begin{bmatrix} 0 & 4 \\ \text{-2} & 1 \end{bmatrix}, \text{ and thus } C_{11} = 8 \\
\boldsymbol{M}_{12} &= \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \text{ and } C_{12} = -1 \\
\boldsymbol{M}_{13} &= \begin{bmatrix} 1 & 0 \\ 0 & \text{-2} \end{bmatrix}, \text{ and } C_{13} = -2 \\
\boldsymbol{M}_{21} &= \begin{bmatrix} 2 & 0 \\ \text{-2} & 1 \end{bmatrix}, \text{ and } C_{21} = -2 \\
\boldsymbol{M}_{22} &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } C_{22} = 0 \\
\boldsymbol{M}_{23} &= \begin{bmatrix} 0 & 2 \\ 0 & \text{-2} \end{bmatrix}, \text{ and } C_{23} = 0 \\
\boldsymbol{M}_{31} &= \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \text{ and } C_{31} = 8 \\
\boldsymbol{M}_{32} &= \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix}, \text{ and } C_{32} = 0 \\
\boldsymbol{M}_{33} &= \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \text{ and } C_{33} = -2
\end{aligned}$$

Therefore, by Lemma 7, we have:

$$\begin{aligned}
\boldsymbol{A}^{-1} &= \frac{1}{det(\boldsymbol{A})} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}. \\
&= -\frac{1}{2} \begin{bmatrix} 8 & -2 & 8 \\ -1 & 0 & 0 \\ -2 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -4 & 1 & -4 \\ 1/2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

□