Lecture Notes: Arc Lengths

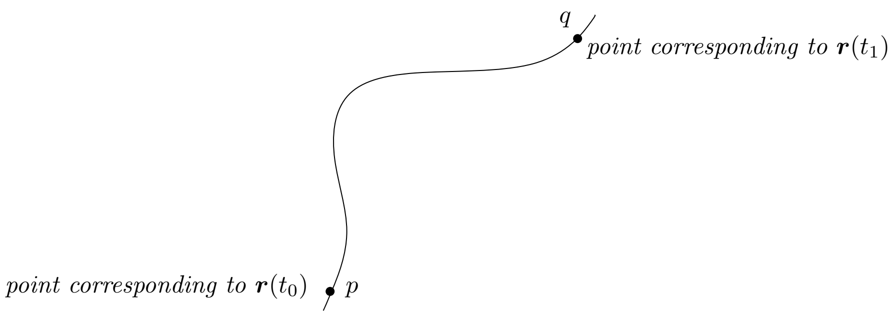
Yufei Tao
Department of Computer Science and Engineering
Chinese University of Hong Kong
taoyf@cse.cuhk.edu.hk

1 Definition of Arc Lengths

Recall that a curve in $\mathbb{R}^d$ can be represented as a vector function $\boldsymbol{r}(t) = [x_1(t), x_2(t), ..., x_d(t)]$, where $x_1(t), x_2(t), ..., x_d(t)$ give the coordinates of the point on the curve corresponding to a value of t. If we take a continuous portion of the curve, we get an *arc*, which is formally defined as:

Definition 1. *Given a curve $\boldsymbol{r}(t)$, an* **arc** *of the curve is $\{\boldsymbol{r}(t) \mid t_0 \le t \le t_1\}$ where t_0 and t_1 are real values.*

It is worth mentioning that the arc as defined above is sometimes also referred to as "*the curve from t_0 to t_1*" or as "*the curve from point $\boldsymbol{r}(t_0)$ to point $\boldsymbol{r}(t_1)$*". In the example below, the curve/arc from t_0 to t_1 is the part of the curve between p and q.

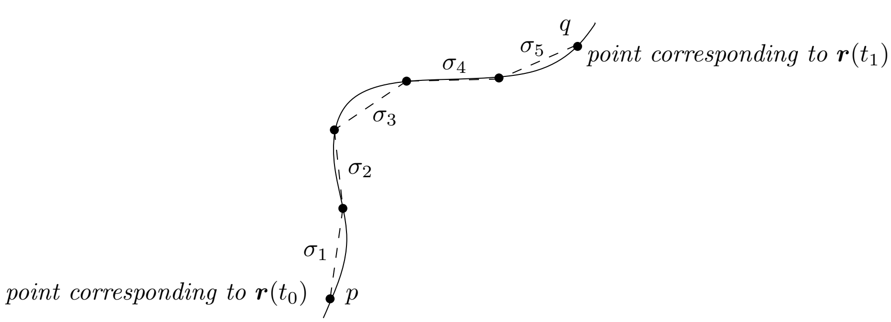


Intuitively, an arc should have a "length", which we formalize below as a limit:

Definition 2. *Let C be an arc given by $\boldsymbol{r}(t)$ with t ranging from t_0 to t_1. Evenly divide the interval $[t_0, t_1]$ by inserting $n+1$ break points $\tau_0, \tau_1, \tau_2, ..., \tau_n$ where $\tau_0 = t_0$ and $\tau_i - \tau_{i-1} = (t_1 - t_0)/n$ for each $i \in [1, n]$. Define σ_i to be the straight line segment connecting the points $\boldsymbol{r}(\tau_{i-1})$ and $\boldsymbol{r}(\tau_i)$, and denote by $|\sigma_i|$ the length of σ_i. Then, if the following limit exists:*

$$\lim_{n \to \infty} \sum_{i=1}^{n} |\sigma_i| \tag{1}$$

we say that the limit is the **length** *of C.*



The figure above shows an example with $n = 5$. Note how we approximate the length of the curve by the total length of a sequence of segments.

In this course, we will be interested mainly in *smooth curves*. Intuitively, these are curves that (i) do not degenerate into a point, and (ii) do not have "corners" (e.g., the boundary of a triangle is not smooth). Mathematically, we formalize the notion as follows:

Definition 3. *Let C be a curve given by $\boldsymbol{r}(t)$ with t ranging from t_0 to t_1. C is* **smooth** *if (i) $\boldsymbol{r}'(t)$ is continuous in $[t_0, t_1]$, and (ii) $\boldsymbol{r}'(t) \neq \mathbf{0}$ at any $t \in [t_0, t_1]$.*

We will state without proof the following lemma:

Lemma 1. *Let C be as described in Definition 2. If C is smooth, then the limit* (1) *always exists.*

2 Computing Arc Lengths

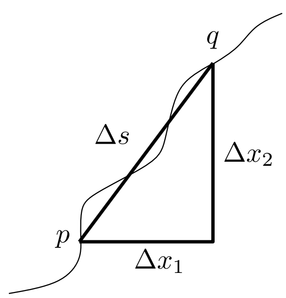
Consider a curve given by the vector function $\boldsymbol{r}(t)$. Fix a real value t_0, and consider the arc C from t_0 to t. Note that C extends as t grows, which means that the length s of C is a function of t.

The following is an important lemma:

Lemma 2. *If C is smooth, then it holds that:*

$$\frac{d(s(t))}{dt} = \sqrt{\sum_{i=1}^{d} \left(\frac{d(x_i(t))}{dt}\right)^2}.$$

We will not present a complete proof of the lemma, but the following discussion will point out the main ideas. Consider the figure below in 2d space. Imagine that we increase t by a tiny amount Δt. By doing so, we have traveled on the curve a little from point p to point q. Δx_1 and Δx_2 give the coordinate differences of p and q on the two dimensions, respectively. When Δt is extremely small, the length Δs of the curve from p to q should be very close to the length of the segment connecting p and q, that is, $\Delta s \approx \sqrt{(\Delta x_1)^2 + (\Delta x_2)^2}$, which gives $\frac{\Delta s}{\Delta t} \approx \sqrt{(\frac{\Delta x_1}{\Delta t})^2 + (\frac{\Delta x_2}{\Delta t})^2}$.



Now fix another real value $t_1 \geq t_0$. Denote by L the length of the arc from t_0 to t_1. We can calculate L as follows:

$$\begin{aligned} L &= \int_0^L ds \\ &= \int_{t_0}^{t_1} \frac{ds}{dt} dt \\ &= \int_{t_0}^{t_1} \sqrt{\sum_{i=1}^{d} \left(\frac{d(x_i(t))}{dt}\right)^2} \, dt. \end{aligned}$$

Example 1. Consider the circle $x^2 + y^2 = 1$. Let p be the point $(1, 0)$ and q the point $(-1, 0)$. Let C be the arc of the circle from p to q. How to calculate the length of C?

First of all, we need to represent the circle using a single parameter. One way of doing so is to define:

$$\begin{aligned} x(t) &= \cos(t) \\ y(t) &= \sin(t). \end{aligned}$$

Then C is essentially the curve from $t = 0$ (point p) to $t = \pi$ (point q). Hence, the length of C is given by:

$$\begin{aligned} \int_0^\pi \frac{ds}{dt}\,dt &= \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ &= \int_0^\pi \sqrt{(-\sin(t))^2 + (\cos(t))^2}\,dt \\ &= \int_0^\pi 1\,dt = \pi. \end{aligned}$$

$\square$

Example 2. Consider the helix $\boldsymbol{r}(t) = [x(t), y(t), z(t)]$ where

$$\begin{aligned} x(t) &= \cos(t) \\ y(t) &= \sin(t) \\ z(t) &= t. \end{aligned}$$

The length of the arc from $t = 0$ to $t = \pi$ is:

$$\begin{aligned} \int_0^\pi \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt &= \int_0^\pi \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1^2}\,dt \\ &= \sqrt{2}\int_0^\pi dt \\ &= \sqrt{2}\pi. \end{aligned}$$

$\square$