Exercises: Orthogonal and Symmetric Matrices

Problem 1. Consider the following set S of column vectors:

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\}$$

Find all the possible $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that makes S an orthogonal set.

Problem 2. Consider the following matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & x \\ 0 & \cos\theta & y \\ 0 & \sin\theta & z \end{bmatrix}$$

Find all the possible $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that makes $\boldsymbol{A}$ orthogonal.

Problem 3. Prove: if matrix $\boldsymbol{A}$ is orthogonal, then its determinants must be either 1 or -1.

Problem 4. Prove: if matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are both orthogonal, then $\boldsymbol{AB}$ is also orthogonal.

Problem 5. Prove: if an $n \times n$ matrix $\boldsymbol{A}$ is orthogonal, then (i) $\boldsymbol{A}^{-1}$ definitely exists, and (ii) $\boldsymbol{A}^{-1}$ must also be orthogonal.

Problem 6. Diagonalize the following matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

into $\boldsymbol{QBQ}^{-1}$ where $\boldsymbol{B}$ is a diagonal matrix, and $\boldsymbol{Q}$ is an orthogonal matrix. You need to give the details of only $\boldsymbol{Q}$ and $\boldsymbol{B}$, namely, you do not need to give the details of $\boldsymbol{Q}^{-1}$.

Problem 7. Suppose that an $n \times n$ matrix $\boldsymbol{A}$ can be computed as $\boldsymbol{QBQ}^{-1}$ where $\boldsymbol{Q}$ is an $n \times n$ orthogonal matrix, and $\boldsymbol{B}$ is an $n \times n$ diagonal matrix. Prove: $\boldsymbol{A}$ is a symmetric matrix.