Exercises: Orthogonal and Symmetric Matrices

Problem 1. Consider the following set S of column vectors:

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\}$$

Find all the possible $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that makes S an orthogonal set.

Solution. For S to be orthogonal, the vectors in S must be mutually orthogonal to each other. We therefore have:

$$\begin{aligned} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= 0 \\ \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= 0 \end{aligned}$$

which gives the following set of equations on variables x, y, z:

$$\begin{aligned} x &= 0 \\ (\cos\theta)y + (\sin\theta)z &= 0. \end{aligned}$$

The set of solutions $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is:

$$\left\{ \begin{bmatrix} 0 \\ -\frac{\sin\theta}{\cos\theta}t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

Problem 2. Consider the following matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & x \\ 0 & \cos\theta & y \\ 0 & \sin\theta & z \end{bmatrix}$$

Find all the possible $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that makes $\boldsymbol{A}$ orthogonal.

Solution. Recall that $\boldsymbol{A}$ is orthogonal if and only if both conditions below are satisfied:

- All column vectors are mutually orthogonal.
- All column vectors have unit length.

In Problem 1, we have already obtained the set of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying the first bullet:

$$\left\{ \begin{bmatrix} 0 \\ -\frac{\sin\theta}{\cos\theta}t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

To satisfy the second bullet, we need:

$$\begin{aligned} x^2 + y^2 + z^2 &= 1 \Rightarrow \\ \left(-\frac{\sin\theta}{\cos\theta}t\right)^2 + t^2 &= 1 \Rightarrow \\ t^2 &= (\cos\theta)^2 \end{aligned} \tag{1}$$

which means that $t = \cos\theta$ or $t = -\cos\theta$. Hence, there are only two $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that can make $\boldsymbol{A}$ orthogonal:

$$\begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix}, \begin{bmatrix} 0 \\ \sin\theta \\ -\cos\theta \end{bmatrix}$$

Problem 3. Prove: if matrix $\boldsymbol{A}$ is orthogonal, then its determinants must be either 1 or -1.

Solution.

$$\begin{aligned} det(\boldsymbol{I}) &= 1 \Rightarrow \\ det(\boldsymbol{A}\boldsymbol{A}^{-1}) &= 1 \Rightarrow \\ (\text{by } \boldsymbol{A}^{-1} = \boldsymbol{A}^T) \quad det(\boldsymbol{A}\boldsymbol{A}^T) &= 1 \Rightarrow \\ det(\boldsymbol{A}) \cdot det(\boldsymbol{A}^T) &= 1 \Rightarrow \\ (\text{as } det(\boldsymbol{A}) = det(\boldsymbol{A}^T)) \quad det(\boldsymbol{A}) \cdot det(A) &= 1 \Rightarrow \end{aligned}$$

which completes the proof.

Problem 4. Prove: if matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are both orthogonal, then $\boldsymbol{AB}$ is also orthogonal.

Solution. It suffices to prove that

$$\begin{aligned} (\boldsymbol{AB})(\boldsymbol{AB})^T &= \boldsymbol{I} \Leftrightarrow \\ (\boldsymbol{AB})(\boldsymbol{B}^T\boldsymbol{A}^T) &= \boldsymbol{I} \end{aligned}$$

Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are both orthogonal, we know: $\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I}$ and $\boldsymbol{B}\boldsymbol{B}^T = \boldsymbol{I}$. Therefore, $\boldsymbol{A}(\boldsymbol{B}\boldsymbol{B}^T)\boldsymbol{A}^T = \boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I}$.

Problem 5. Prove: if an $n \times n$ matrix $\boldsymbol{A}$ is orthogonal, then (i) $\boldsymbol{A}^{-1}$ definitely exists, and (ii) $\boldsymbol{A}^{-1}$ must also be orthogonal.

Solution. Since $\boldsymbol{A}$ is orthogonal, its row vectors form an orthogonal set, which therefore is linearly independent. This means that $\boldsymbol{A}$ has rank n, meaning that $\boldsymbol{A}^{-1}$ definitely exists.

To prove that $\boldsymbol{A}^{-1}$ is orthogonal, we need to prove $\boldsymbol{A}^{-1} \cdot (\boldsymbol{A}^{-1})^T = \boldsymbol{I}$. For this purpose, note that since $\boldsymbol{A}$ is orthogonal, we have $\boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I}$, namely $\boldsymbol{A}^{-1} = \boldsymbol{A}^T$. Equipped with this fact, we can show $\boldsymbol{A}^{-1} \cdot (\boldsymbol{A}^{-1})^T = \boldsymbol{I}$ as follows:

$$\begin{aligned} \boldsymbol{I}^T &= \boldsymbol{I} \Rightarrow \\ (\boldsymbol{A}^{-1}\boldsymbol{A})^T &= \boldsymbol{I} \Rightarrow \\ \boldsymbol{A}^T(\boldsymbol{A}^{-1})^T &= \boldsymbol{I} \Rightarrow \\ \boldsymbol{A}^{-1} \cdot (\boldsymbol{A}^{-1})^T &= \boldsymbol{I} \end{aligned} \tag{2}$$

which completes the proof.

Problem 6. Diagonalize the following matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

into $\boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^{-1}$ where $\boldsymbol{B}$ is a diagonal matrix, and $\boldsymbol{Q}$ is an orthogonal matrix. You need to give the details of only $\boldsymbol{Q}$ and $\boldsymbol{B}$, namely, you do not need to give the details of $\boldsymbol{Q}^{-1}$.

Solution. We aim to obtain three eigenvectors of $\boldsymbol{A}$ — denote them as $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ respectively — that are mutually orthogonal to each other and have lengths 1.

To start with, find the eigenvalues of $\boldsymbol{A}$: $\lambda_1 = 1$ and $\lambda_2 = -1$.

Now, obtain the eigenspace of λ_1:

$$\left\{ \begin{bmatrix} u \\ u \\ v \end{bmatrix} \;\middle|\; u, v \in \mathbb{R} \right\}.$$

This set has dimension 2. We will first take from the set two eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ that are orthogonal to each other. For this purpose, first set $\boldsymbol{x}_1$ to an arbitrary non-zero vector, e.g., $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Regarding $\boldsymbol{x}_2 = \begin{bmatrix} u \\ u \\ v \end{bmatrix}$, we ensure orthogonality between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ by requiring their dot product to be 0:

$$\begin{aligned} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} u \\ u \\ v \end{bmatrix} &= 0 \Rightarrow \\ u + u &= 0 \Rightarrow \\ u &= 0. \end{aligned}$$

Note that there is no constraint on v. We can set v to be any value such that $\boldsymbol{x}_2$ is not a zero-vector, e.g., $v = 1$ which gives $\boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Finally, normalize $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ to have length 1, which gives:

$\boldsymbol{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Next, obtain the eigenspace of λ_2:

$$\left\{ \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}.$$

This set has dimension 1. Take an arbitrary eigenvector from the set, e.g., $\boldsymbol{x}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. Normalizing this vector to have length 1 gives $\boldsymbol{v}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$.

Therefore:

$$\begin{aligned} \boldsymbol{Q} &= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \\ \boldsymbol{B} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \end{aligned}$$

Problem 7. Suppose that an $n \times n$ matrix $\boldsymbol{A}$ can be computed as $\boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^{-1}$ where $\boldsymbol{Q}$ is an $n \times n$ orthogonal matrix, and $\boldsymbol{B}$ is an $n \times n$ diagonal matrix. Prove: $\boldsymbol{A}$ is a symmetric matrix.

Solution. We aim to prove that $\boldsymbol{A} = \boldsymbol{A}^T$. Towards this purpose, we compute $\boldsymbol{A}^T$ as follows:

$$\begin{aligned} \boldsymbol{A}^T &= (\boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^{-1})^T \;\Rightarrow \\ \boldsymbol{A}^T &= (\boldsymbol{Q}^{-1})^T \boldsymbol{B}^T \boldsymbol{Q}^T \end{aligned} \tag{3}$$

Since $\boldsymbol{Q}$ is an orthogonal matrix, we have: $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^T$. Hence:

$$(3) = (\boldsymbol{Q}^T)^T \boldsymbol{B}^T \boldsymbol{Q}^{-1} = \boldsymbol{Q}\boldsymbol{B}\boldsymbol{Q}^{-1} = \boldsymbol{A}.$$

This completes the proof.