Exercises: Similarity Transformation

Problem 1. Diagonalize the following matrix:

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

Solution. Matrix $\boldsymbol{A}$ has two eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$. Since (i) $\boldsymbol{A}$ is a 2×2 matrix and (ii) it has 2 distinct eigenvalues, we can apply the diagonalization method we discussed in class. Specifically, we obtain an arbitrary eigenvector $\boldsymbol{v_1}$ of λ_1, say $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and, and an arbitrary eigenvector $\boldsymbol{v_2}$ of λ_2, say $\boldsymbol{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Then, we form:

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

by using $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ as the first and second columns, respectively. $\boldsymbol{Q}$ has the inverse:

$$\boldsymbol{Q}^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$$

We thus obtain the following diagonalization of $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{Q}\, diag[3, 2]\, \boldsymbol{Q}^{-1}.$$

Problem 2. Consider again the matrix $\boldsymbol{A}$ in Problem 5. Calculate $\boldsymbol{A}^t$ for any integer $t \geq 1$.

Solution. We already know that $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{Q}\, diag[3, 2]\, \boldsymbol{Q}^{-1}.$$

Hence:

$$\begin{aligned} \boldsymbol{A}^t &= \boldsymbol{Q}\, diag[3^t, 2^t]\, \boldsymbol{Q}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 3^t & 0 \\ 0 & 2^t \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -3^t + 2^{t+1} & -3^t + 2^t \\ 2 \times 3^t - 2^{t+1} & 2 \times 3^t - 2^t \end{bmatrix} \end{aligned}$$

Problem 3. Diagonalize the matrix $\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Solution. Recall that all symmetric matrices are diagonalizable. $\boldsymbol{A}$ is a 3×3 matrix. The key is to find three linearly independent eigenvectors.

From the solution of Problem 1, we know that $\boldsymbol{A}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. $EigenSpace(\lambda_1)$ includes all $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying

$$\begin{aligned} x_1 &= u \\ x_2 &= v \\ x_3 &= u \end{aligned}$$

for any $u, v \in \mathbb{R}$. The vector space $EigenSpace(\lambda_1)$ has dimension 2 with a basis $\{\boldsymbol{v_1}, \boldsymbol{v_2}\}$ where $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (given by $u = 1, v = 0$) and $\boldsymbol{v_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (given by $u = 0, v = 1$).

Similarly, $EigenSpace(\lambda_2)$ includes all $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ satisfying

$$\begin{aligned} x_1 &= u \\ x_2 &= 0 \\ x_3 &= -u \end{aligned}$$

for any $u \in \mathbb{R}$. The vector space $EigenSpace(\lambda_2)$ has dimension 1 with a basis $\{\boldsymbol{v_3}\}$ where $\boldsymbol{v_3} = \begin{bmatrix} 1 \\ 0 \\ \text{-1} \end{bmatrix}$ (given by $u = 1$).

So far, we have obtained three linearly independent eigenvectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$ of $\boldsymbol{A}$. We can then apply the diagonalization method exemplified in Problem 5 to diagonalize $\boldsymbol{A}$. Specifically, we form:

$$\boldsymbol{Q} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$\boldsymbol{Q}$ has the inverse:

$$\boldsymbol{Q}^{-1} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$

We thus obtain the following diagonalization of $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{Q}\, diag[1, 1, -1]\, \boldsymbol{Q}^{-1}.$$

Problem 4. Suppose that matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are similar to each other, namely, there exists $\boldsymbol{P}$ such that $\boldsymbol{A} = \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P}$. Prove: if $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ under eigenvalue λ, then $\boldsymbol{P}\boldsymbol{x}$ is an eigenvector of $\boldsymbol{B}$ under eigenvalue λ.

Solution. By definition of similarity, we know $\boldsymbol{A} = \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P}$. We proved in the lecture that λ must also be an eigenvalue of $\boldsymbol{B}$. Since $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ under λ, we know:

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \lambda \boldsymbol{x} \Rightarrow \\ \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P}\boldsymbol{x} &= \lambda \boldsymbol{x} \Rightarrow \\ \boldsymbol{B}(\boldsymbol{P}\boldsymbol{x}) &= \lambda(\boldsymbol{P}\boldsymbol{x}) \end{aligned}$$

which completes the proof.

Problem 5. Suppose that an $n \times n$ matrix $\boldsymbol{A}$ has n linearly independent eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$. Prove: for any $n \times 1$ vector $\boldsymbol{x}$, $\boldsymbol{A}\boldsymbol{x}$ is a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$.

Solution. Assume that $\boldsymbol{v}_i$ ($i \in [1, k]$) is an eigenvector of $\boldsymbol{A}$ under eigenvalue λ_i. We have $\boldsymbol{A}\boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$. Since $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$ are linearly independent, we know that $\boldsymbol{x}$ must be a linear combination $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n$. Namely, there exist $c_1, ..., c_n$ such that

$$\begin{aligned}
\boldsymbol{x} &= c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + ... + c_n\boldsymbol{v}_n \Rightarrow \\
\boldsymbol{A}\boldsymbol{x} &= c_1\boldsymbol{A}\boldsymbol{v}_1 + c_2\boldsymbol{A}\boldsymbol{v}_2 + ... + c_n\boldsymbol{A}\boldsymbol{v}_n \Rightarrow \\
\boldsymbol{A}\boldsymbol{x} &= c_1\lambda_1\boldsymbol{v}_1 + c_2\lambda_2\boldsymbol{v}_2 + ... + c_n\lambda_n\boldsymbol{v}_n.
\end{aligned}$$

which completes the proof.

Problem 6. Prove or disprove: if an $n \times n$ matrix $\boldsymbol{A}$ has rank n, then it must have n independent eigenvectors.

Solution. False.

Consider $n = 2$ and $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. It has only one distinct eigenvalue 1. Thus, any eigenvector $\boldsymbol{v}$ of $\boldsymbol{A}$ must satisfy:

$$\begin{aligned}
(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{x} &= 0 \Rightarrow \\
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{x} &= 0
\end{aligned}$$

Thus, any eigenvector of $\boldsymbol{A}$ must have the form $\left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \mid t \in \mathbb{R}, t \neq 0 \right\}$. This set of vectors has a dimension of 1.

Problem 7. Prove that $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

Solution. $\boldsymbol{A}$ has two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$. Let $\boldsymbol{v}_1$ be an eigenvector of λ_1. $\boldsymbol{v}_1$ must satisfy:

$$\begin{aligned}
(\boldsymbol{A} - \lambda_1\boldsymbol{I})\boldsymbol{v}_1 &= 0 \Rightarrow \\
\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\boldsymbol{v}_1 &= 0 \Rightarrow
\end{aligned}$$

Hence, the set of eigenvectors of λ_1 is:

$$\left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \mid t \in \mathbb{R}, t \neq 0 \right\}$$

This set has dimension 1.

Let $\boldsymbol{v}_2$ be an eigenvector of λ_2. $\boldsymbol{v}_2$ must satisfy:

$$\begin{aligned}(\boldsymbol{A}-\lambda_1\boldsymbol{I})\boldsymbol{v}_2 &= 0 \Rightarrow \\ \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\boldsymbol{v}_2 &= 0 \Rightarrow\end{aligned}$$

Hence, the set of eigenvectors of λ_2 is:

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix} \Big| \, t \in \mathbb{R}, t \neq 0 \right\}$$

This set also has dimension 1.

It thus follows that the largest number of linearly independent eigenvectors of $\boldsymbol{A}$ is $1+1=2$. Therefore, $\boldsymbol{A}$ is not diagonalizable.

Problem 8. Let $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ be three $n \times n$ matrices for some integer n. Prove that if $\boldsymbol{A}$ is similar to $\boldsymbol{B}$ and $\boldsymbol{B}$ is similar to $\boldsymbol{C}$, then $\boldsymbol{A}$ is similar to $\boldsymbol{C}$.

Solution. From the fact that $\boldsymbol{A}$ is similar to $\boldsymbol{B}$ and $\boldsymbol{B}$ is similar to $\boldsymbol{C}$, we know:

$$\boldsymbol{A} = \boldsymbol{P}^{-1}\boldsymbol{B}\boldsymbol{P}$$

and

$$\boldsymbol{B} = \boldsymbol{Q}^{-1}\boldsymbol{C}\boldsymbol{Q}.$$

Hence:

$$\boldsymbol{A} = \boldsymbol{P}^{-1}\boldsymbol{Q}^{-1}\boldsymbol{B}\boldsymbol{Q}\boldsymbol{P} = (\boldsymbol{Q}\boldsymbol{P})^{-1}\boldsymbol{B}(\boldsymbol{Q}\boldsymbol{P})$$

which completes the proof.

Problem 9. Decide whether

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

is similar to

$$\boldsymbol{B} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}.$$

Solution 1. From Problem 1, we know that $\boldsymbol{A}$ has distinct eigenvalues 3 and 2. Hence, $\boldsymbol{A}$ is similar to the diagonal matrix $diag[3,2]$. On the other hand, $\boldsymbol{B}$ clearly also has eigenvalues 3 and 2, and thus, is also similar to $diag[3,2]$. From the result of Problem 8, we know that $\boldsymbol{A}$ is similar to $\boldsymbol{B}$.

Solution 2. We will try to find an invertible matrix $\boldsymbol{P} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ that makes $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{B}\boldsymbol{P}^{-1}$ hold. This is equivalent to $\boldsymbol{A}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{B}$. Hence:

$$\begin{aligned} \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} &= \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \Rightarrow \\ \begin{bmatrix} x-z & y-w \\ 2x+4z & 2y+4w \end{bmatrix} &= \begin{bmatrix} 3x & x+2y \\ 3z & z+2w \end{bmatrix} \end{aligned}$$

This gives the following equation set:

$$\begin{aligned} x - z &= 3x \\ y - w &= x + 2y \\ 2x + 4z &= 3z \\ 2y + 4w &= z + 2w \end{aligned}$$

You can verify that the set of solutions $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ is $\left\{ \begin{bmatrix} -u/2 \\ u/2 - v \\ u \\ v \end{bmatrix} \mid u \in \mathbb{R}, v \in \mathbb{R} \right\}$.

Let us try $u = 2, v = 0$. This gives $\boldsymbol{P} = \begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}$. Since $det(\boldsymbol{P}) \neq 0$, we know that $\boldsymbol{P}$ is invertible. We can now conclude that $\boldsymbol{A}$ is similar to $\boldsymbol{B}$.