Exercises: Line Integral by Length

Problem 1. Let C be the curve from point $p(0,0)$ to point $q(1,1)$ on the parabola $y = x^2$. Calculate $\int_C x\,ds$.

Solution: First, write C into its parametric form: $\boldsymbol{r}(t) = [x(t), y(t)]$ where $x(t) = t$, and $y(t) = t^2$. Points p and q are given by $t = 0$ and 1, respectively. Thus:

$$\begin{aligned}
\int_C x\,ds &= \int_0^1 x(t)\frac{ds}{dt}dt \\
&= \int_0^1 x(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_0^1 t\sqrt{1+4t^2}\,dt \\
&= \frac{1}{12}(1+4t^2)^{3/2}\Big|_0^1 = \frac{5\sqrt{5}-1}{12}.
\end{aligned}$$

Problem 2. Let C be the line segment from point $p(1,2,3)$ to point $q(8,7,6)$. Calculate $\int_C x+z^2\,ds$.

Solution: Vector $\boldsymbol{q}-\boldsymbol{p} = [8,7,6]-[1,2,3] = [7,5,3]$ gives the direction of the line segment. Hence, C can be written into its parametric form: $\boldsymbol{r}(t) = [x(t), y(t), z(t)]$ where $x(t) = 1+7t$, $y(t) = 2+5t$, and $z(t) = 3+3t$. Points p and q are given by $t = 0$ and $t = 1$, respectively. Thus:

$$\begin{aligned}
\int_C x+z^2\,ds &= \int_0^1 (x(t)+(z(t))^2)\,\frac{ds}{dt}dt \\
&= \int_0^1 (1+7t+(3+3t)^2)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \\
&= \int_0^1 (10+25t+9t^2)\sqrt{7^2+5^2+3^2}dt \\
&= \sqrt{83}\int_0^1 (10+25t+9t^2)dt \\
&= \frac{51\sqrt{83}}{2}.
\end{aligned}$$

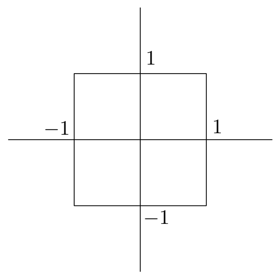
Problem 3. Let C be the circle $x^2+y^2 = 1$. Calculate $\int_C y\,ds$.

Solution: Introduce the parametric form of C: $\boldsymbol{r}(t) = [x(t), y(t)]$ where $x(t) = \cos(t)$ and $y(t) = \sin(t)$. Pick an arbitrary point on C, e.g., $p(1,0)$. Let $q = p$, i.e., another copy of the same point.

View p as being given by $t = 0$, and q as being given by $t = 2\pi$.

$$\begin{aligned}
\int_C y\,ds &= \int_0^{2\pi} \sin(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_0^{2\pi} \sin(t)\sqrt{(-\sin(t))^2 + (\cos(t))^2}\,dt \\
&= \int_0^{2\pi} \sin(t)\,dt \\
&= -\cos(t)\Big|_0^{2\pi} = 0.
\end{aligned}$$

Problem 4. Let C be the boundary of the square shown below:



Calculate $\int_C y\,ds$.

Solution. C is a piecewise-smooth curve. Define:

- C_1: the bottom edge of C.
- C_2: the right edge of C.
- C_3: the top edge of C.
- C_4: the left edge of C.

We have:

$$\int_C y\,ds = \int_{C_1} y\,ds + \int_{C_2} y\,ds + \int_{C_3} y\,ds + \int_{C_4} y\,ds$$

Next, we compute each integral on the right hand side in turn:

$$\int_{C_1} y\,ds = -\int_{C_1} ds = -2.$$

C_2 can be represented as $\{[x(t) = 1, y(t) = t] \mid -1 \leq t \leq 1\}$.

$$\begin{aligned}
\int_{C_2} y\,ds &= \int_{-1}^{1} y \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_{-1}^{1} t \cdot \sqrt{0+1}\,dt = \frac{t^2}{2}\Big|_{-1}^{1} = 0
\end{aligned}$$

Similarly, we can get:

$$\begin{aligned}\int_{C_3} y\,ds &= 2\\ \int_{C_4} y\,ds &= 0.\end{aligned}$$

Therefore, $\int_C y\,ds = 0$.

Remark. Interestingly, $\int_C y\,ds = 0$ can also be inferred directly from the definition of line integral by arc length. Hint: break each edge into subintervals, and argue that each subinterval will get "canceled" by another subinterval in the summation that defines the line integral.

Problem 5. Let C be the intersection of two surfaces: sphere $x^2 + y^2 + z^2 = 3$ and plane $x = y$. Calculate $\int_C x^2 ds$.

Solution: Observe first that the intersection is a circle, which is a closed curve. Introduce $x(t) = y(t) = \frac{\sqrt{3}}{\sqrt{2}}\cos(t)$ and $z(t) = \sqrt{3}\sin(t)$. Pick a point on C by setting $t = 0$, which gives $p(\sqrt{3/2}, \sqrt{3/2}, 0)$. What is the smallest t that will give the same p? Clearly, the answer is $t = 2\pi$. Define $q = p$, and view q as being given by $t = 2\pi$. Thus, C can be regarded as the trail of $[x(t), y(t), z(t)]$ as t grows from 0 to 2π.

$$\begin{aligned}\int_C x^2\,ds &= \int_0^{2\pi} \frac{3}{2}(\cos(t))^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt\\ &= \int_0^{2\pi} \frac{3}{2}(\cos(t))^2 \sqrt{\left(-\frac{\sqrt{3}}{\sqrt{2}}\sin(t)\right)^2 + \left(-\frac{\sqrt{3}}{\sqrt{2}}\sin(t)\right)^2 + \left(\sqrt{3}\cos(t)\right)^2}\,dt\\ &= \frac{3\sqrt{3}}{2}\int_0^{2\pi} (\cos(t))^2\,dt\\ &= \frac{3\sqrt{3}}{2}\left(\frac{t}{2} + \frac{\sin(2t)}{4}\right)\Big|_0^{2\pi}\\ &= \frac{3\sqrt{3}}{2}\pi.\end{aligned}$$